Week 9: Exam Week: Timed Mini-Set

Handout · Thu Dec 3 / Fri Dec 4

Timed set: 35 minutes, work alone, pens down at the bell. Simulate the real thing. Read all three problems first, rank them, and start with the one you understand best. Write *full solutions* to as many as you can, on separate sheets, in the form you would hand to a grader: the answer stated up front, full sentences, every key step justified, the tool named as you use it. A solved problem with no write-up scores 0; one complete, clean solution is a good session. No collaboration, no phones, no notes: only what you would have on Saturday.

Problem 1. ⋆⋆ *(Putnam 2013 A1)* Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

Problem 2. ⋆⋆ *(Putnam 2024 A1)* Determine all positive integers n for which there exist positive integers a, b, and c satisfying $2a^n + 3b^n = 4c^n$.

Problem 3. ⋆⋆ *(Putnam 2020 A2)* Let k be a nonnegative integer. Evaluate $\sum_{j=0}^{k} 2^{k-j}\binom{k+j}{j}$.

Take-home: Problems 4–6

Problem 4. ⋆⋆⋆ *(Putnam 2021 A3)* Determine all positive integers N for which the sphere $x^2 + y^2 + z^2 = N$ has an inscribed regular tetrahedron whose vertices have integer coordinates.

Problem 5. ⋆⋆⋆ *(Putnam 2023 A4)* Let $v_1, \ldots, v_{12}$ be unit vectors in $\mathbb{R}^3$ from the origin to the vertices of a regular icosahedron. Show that for every vector $v \in \mathbb{R}^3$ and every $\varepsilon > 0$, there exist integers $a_1, \ldots, a_{12}$ such that $\|a_1v_1 + \cdots + a_{12}v_{12} - v\| < \varepsilon$.

Problem 6. ⋆⋆⋆ *(Putnam 2019 A3)* Given real numbers $b_0, b_1, \ldots, b_{2019}$ with $b_{2019} \neq 0$, let $z_1, z_2, \ldots, z_{2019}$ be the roots in the complex plane of the polynomial $P(z) = \sum_{k=0}^{2019} b_k z^k$. Let $\mu = \dfrac{|z_1| + \cdots + |z_{2019}|}{2019}$ be the average of the distances from $z_1, \ldots, z_{2019}$ to the origin. Determine the largest constant M such that $\mu \geq M$ for all choices of $b_0, b_1, \ldots, b_{2019}$ that satisfy $1 \leq b_0 < b_1 < b_2 < \cdots < b_{2019} \leq 2019$.

Problems 1–3 are the timed in-session set; 4–6 are take-home. Solutions will be posted after Friday's session.

Good luck on Saturday. Sleep on Friday, eat on Saturday, state the answer first, and write the key step. One complete solution is a real achievement, and you are ready for it.