Week 8: Probability, Expectation, and Games

Lecture notes · Thu Nov 19 / Fri Nov 20

Toolbox: expectation without computing distributions, and games without computing trees.

1. **Linearity.** $\mathbb{E}[X_1 + \cdots + X_m] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_m]$ for *any* random variables, dependent or not. Use it through **indicators**: if X counts how many of the events $A_1, \ldots, A_m$ occur, then $X = \sum \mathbf{1}_{A_i}$ and $\mathbb{E}[X] = \sum_i \mathbb{P}(A_i)$. Choosing what to count is the whole art.

2. **Existence from averages.** Some outcome has $X \geq \mathbb{E}[X]$ and some has $X \leq \mathbb{E}[X]$; if X is a nonnegative integer with $\mathbb{E}[X] < 1$, then $\mathbb{P}(X = 0) > 0$. To prove "some object has property P," pick one at random and show the expected number of defects is < 1.

3. **Symmetry.** Independent continuous random variables are almost surely distinct and equally likely to be in each of the $n!$ orders: $\mathbb{P}(X_1 < \cdots < X_n) = 1/n!$, $\mathbb{P}(X_i = \max\{X_1, \ldots, X_i\}) = 1/i$, and a symmetric function of $(X_1, \ldots, X_n)$ is independent of the order.

4. **Conditioning.** $\mathbb{E}[X] = \mathbb{E}\big[\mathbb{E}[X \mid Y]\big]$, and $\mathbb{P}(A) = \sum_j \mathbb{P}(A \mid B_j)\mathbb{P}(B_j)$ for a partition $\{B_j\}$. For a random process $a_0, a_1, \ldots$ condition on one step: if $\mathbb{E}[a_{k+1} \mid a_k]$ is an affine function of a_k, then $e_k = \mathbb{E}[a_k]$ satisfies a linear recursion you can solve exactly.

5. **Tail formulas.** For $X \geq 0$, $\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > t)\,dt$; for integer $X \geq 0$, $\mathbb{E}[X] = \sum_{k \geq 1} \mathbb{P}(X \geq k)$. So $\mathbb{E}[\min(U_1, \ldots, U_i)] = \int_0^1 (1-t)^i\,dt = \frac{1}{i+1}$ for independent uniforms on $[0, 1]$.

6. **Games.** A finite two-player perfect-information game with no draws has exactly one player with a winning strategy (Zermelo). Three ways to find it: *pairing/symmetry* (answer every move with its mirror image; verify the mirror move is legal and not the move just played); *strategy stealing* (if the second player had a winning strategy, the first player could make a harmless move and then use it; contradiction); *invariants* (a set $\mathcal{P}$ of positions such that every move from $\mathcal{P}$ leaves it and from every position outside $\mathcal{P}$ some move enters it; the player who moves *to* $\mathcal{P}$ wins).

Example 1 (Indicators and existence)**.** Every graph with m edges has a bipartite subgraph with at least $m/2$ edges. *Color each vertex red or blue independently with probability $\frac{1}{2}$ each. For an edge e, let $I_e = 1$ if its endpoints get different colors; then $\mathbb{P}(I_e = 1) = \frac{1}{2}$, regardless of what happens at other edges. The number of two-colored edges is $X = \sum_e I_e$, so by linearity $\mathbb{E}[X] = m/2$. Some coloring therefore has $X \geq m/2$; the two-colored edges of that coloring form a bipartite subgraph. Note that the I_e are far from independent (edges sharing a vertex), and it did not matter.*

Example 2 (Conditioning and induction: Putnam 2002 B1)**.** Shanille hits her first free throw, misses her second, and thereafter hits each shot with probability equal to the fraction of shots hit so far. Find the probability she hits exactly 50 of her first 100. *Claim: after $n \geq 2$ shots, the number of hits is uniform on $\{1, \ldots, n-1\}$. For $n = 2$ it is 1 with probability 1. Assume the claim for n and condition on the count H_n after n shots. For $1 \leq h \leq n$,*

$$\mathbb{P}(H_{n+1} = h) = \mathbb{P}(H_n = h-1) \cdot \frac{h-1}{n} + \mathbb{P}(H_n = h) \cdot \frac{n-h}{n} = \frac{1}{n-1} \cdot \frac{(h-1) + (n-h)}{n} = \frac{1}{n},$$

where the terms with $h - 1 = 0$ or $h = n$ vanish because $\mathbb{P}(H_n = 0) = \mathbb{P}(H_n = n) = 0$. So after 100 shots each count in $\{1, \ldots, 99\}$ has probability $1/99$; the answer is $1/99$. The move was "condition on the previous state and let the induction hypothesis do the arithmetic."

Example 3 (Strategy stealing: Chomp)**.** An $m \times n$ chocolate bar ($mn > 1$) has a poisoned square at the bottom left. Players alternately pick a square and eat it together with every square above it, to its right, or both (the whole quadrant); whoever eats the poison loses. The first player has a winning strategy. *The game is finite and has no draws, so one player has a winning strategy. Suppose it is the second player. Let the first player eat only the top-right square. The second player's winning reply eats some square s, producing a position P. But eating s also removes the top-right square, so the first player could have reached P directly by eating s on move one, and from then on could use the second player's winning strategy as her own. Both players cannot have winning strategies, so the first player does. We have no idea what her first move is (for general $m \times n$ nobody does).*

Pitfall. (1) Linearity never needs independence, but products and variances do: $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ and $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ are *claims* to be justified, not defaults. (2) $\mathbb{E}[1/X] \neq 1/\mathbb{E}[X]$ and $\mathbb{E}[\max] \neq \max \mathbb{E}$; when a quantity is not a sum, look for a recursion or a tail formula. (3) Symmetry arguments assume ties have probability 0; say so. (4) Swapping $\mathbb{E}$ and an infinite $\sum$ needs a line; for nonnegative terms, monotone convergence does it. (5) A pairing strategy fails if the paired move is ever illegal or coincides with the move just made; check both. (6) Strategy stealing proves *who* wins, never *how*; do not claim a strategy you have not written down.

How to recognize this technique on the exam. "Find the expected value of" a count or a length is almost always indicators plus linearity: compute one $\mathbb{P}(A_i)$ by symmetry and add. If the quantity is a maximum, a stopping time, or a sum up to a random index, either rewrite it as a sum (of indicators, or of terms $X_i \mathbf{1}_{\{k \geq i\}}$) or condition on the first step and set up a recursion; a limit like $\lim E(n)/n$ signals that the recursion for $\mathbb{E}[a_k]$ closes exactly. "Show that some object exists" with a numerical bound may be asking for a random construction and $\mathbb{E}[\text{defects}] < 1$. For games, ask in order: is there a symmetry to mirror? does an extra move ever hurt (strategy stealing)? is there an invariant, such as a parity or a pairing of the pieces, that one player can restore after every move?

Suggested timing (15 minutes). 0–4: linearity and indicators, Example 1, the "some outcome $\geq$ mean" principle. 4–8: symmetry ($1/i$, $1/n!$) and conditioning, Example 2 as the model of a one-step recursion. 8–12: games: Zermelo's dichotomy, pairing (mention the coin game without solving it), Example 3. 12–15: pitfalls (1), (5), (6), then groups start on Problems 1–4. Remind students this is the last session before Thanksgiving; Week 9 is the timed mini-set.