# Week 8: Probability, Expectation, and Games

Handout · Thu Nov 19 / Fri Nov 20

**Linearity** needs no independence: write the count as a sum of indicators, $X = \sum_i \mathbf{1}_{A_i}$, so $\mathbb{E}[X] = \sum_i \mathbb{P}(A_i)$; some outcome has $X \geq \mathbb{E}[X]$. **Symmetry:** independent continuous variables are equally likely to be in any of the $n!$ orders, so "$X_i$ is the largest of $X_1, \ldots, X_i$" has probability $1/i$. **Conditioning:** $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]]$; for a random process, condition on one step to get a recursion for $\mathbb{E}[a_k]$, or prove a distributional claim by induction. **Games:** in a finite two-player game of perfect information with no draws exactly one player has a winning strategy; look for a *pairing* (mirror every move; check the mirror move is legal), *strategy stealing* (an extra move never hurts, so the first player wins), or an *invariant* one player can always restore.

**Problem 1.** $\star$ *(warm-up)* A permutation $\sigma$ of $\{1, 2, \ldots, n\}$ is chosen uniformly at random.

(a) Find the expected number of fixed points of $\sigma$ (indices $i$ with $\sigma(i) = i$).

(b) Call $i$ a *record* if $\sigma(i) > \sigma(j)$ for every $j < i$ (so $i = 1$ is always a record). Find the expected number of records.

**Problem 2.** $\star$ *(classical)* Two players alternately place identical circular coins on a round table large enough to hold at least one coin. Each coin must lie entirely on the table and not overlap any coin already placed (touching is allowed). A player who cannot place a coin loses. Show that the first player has a winning strategy.

**Problem 3.** $\star\star$ *(Putnam 2022 A4)* Suppose that $X_1, X_2, \ldots$ are real numbers between 0 and 1 that are chosen independently and uniformly at random. Let $S = \sum_{i=1}^{k} X_i/2^i$, where $k$ is the least positive integer such that $X_k < X_{k+1}$, or $k = \infty$ if there is no such integer. Find the expected value of $S$.

**Problem 4.** $\star\star$ *(Putnam 2002 B4)* An integer $n$, unknown to you, has been randomly chosen in the interval $[1, 2002]$ with uniform probability. Your objective is to select $n$ in an *odd* number of guesses. After each incorrect guess, you are informed whether $n$ is higher or lower, and you must guess an integer on your next turn among the numbers that are still feasibly correct. Show that you have a strategy so that the chance of winning is greater than $2/3$.

**Problem 5.** $\star\star\star$ *(Putnam 2023 B3)* A sequence $y_1, y_2, \ldots, y_k$ of real numbers is called *zigzag* if $k = 1$, or if $y_2 - y_1,\ y_3 - y_2,\ \ldots,\ y_k - y_{k-1}$ are nonzero and alternate in sign. Let $X_1, X_2, \ldots, X_n$ be chosen independently from the uniform distribution on $[0, 1]$. Let $a(X_1, X_2, \ldots, X_n)$ be the largest value of $k$ for which there exists an increasing sequence of integers $i_1, i_2, \ldots, i_k$ such that $X_{i_1}, X_{i_2}, \ldots, X_{i_k}$ is zigzag. Find the expected value of $a(X_1, X_2, \ldots, X_n)$ for $n \geq 2$.

**Problem 6.** $\star\star\star$ *(Putnam 2024 B4)* Let $n$ be a positive integer. Set $a_{n,0} = 1$. For $k \geq 0$, choose an integer $m_{n,k}$ uniformly at random from the set $\{1, \ldots, n\}$, and let $a_{n,k+1} = a_{n,k} + 1$ if $m_{n,k} > a_{n,k}$; $a_{n,k+1} = a_{n,k}$ if $m_{n,k} = a_{n,k}$; and $a_{n,k+1} = a_{n,k} - 1$ if $m_{n,k} < a_{n,k}$. Let $E(n)$ be the expected value of $a_{n,n}$. Determine $\lim_{n\to\infty} E(n)/n$.

*Problems 1–4 are the in-session targets; 5–6 are take-home. Solutions will be posted after Friday's session.*