Week 7: Linear Algebra: Determinants, Eigenvalues, Rank, Structured Matrices

Lecture notes · Thu Nov 12 / Fri Nov 13

Toolbox: turn a matrix statement into a scalar statement.

1. **Determinant.** Multilinear and alternating in the rows (and columns): adding a multiple of one row to another changes nothing, a swap flips the sign, two equal rows give 0. Also $\det(AB) = \det A \det B$, $\det A^T = \det A$, $\det(cA) = c^n \det A$, block-triangular $\Rightarrow$ product of the diagonal blocks' determinants, and $\det A = \prod \lambda_i$ over the complex eigenvalues with algebraic multiplicity.
2. **Eigenvalues and trace.** $\operatorname{tr} A = \sum \lambda_i$ and $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. If $p(A) = 0$ for a polynomial p, every eigenvalue is a root of p; the eigenvalues of $p(A)$ are $p(\lambda_i)$. A real matrix has real characteristic polynomial, so nonreal eigenvalues come in conjugate pairs $\lambda, \bar\lambda$ with $\lambda\bar\lambda = |\lambda|^2 > 0$. Symmetric $\Rightarrow$ real eigenvalues; orthogonal ($Q^TQ = I$) $\Rightarrow |\lambda| = 1$, $\det Q = \pm 1$. Cayley–Hamilton: $\chi_A(A) = 0$.
3. **Rank.** $\operatorname{rank} A + \dim\ker A = n$; $\operatorname{rank}(AB) \le \min(\operatorname{rank} A, \operatorname{rank} B)$; $\det A \ne 0 \iff \operatorname{rank} A = n \iff \ker A = 0$. A matrix $M - \lambda I$ of rank r says λ is an eigenvalue of geometric (hence algebraic) multiplicity $\ge n - r$; the trace then often pins down the last eigenvalue.
4. **Structured matrices.** Rank one: uv^T, with $\det(I + uv^T) = 1 + v^Tu$. All-ones J: eigenvalue n on $\mathbf{1}$ and 0 on the hyperplane $\sum x_i = 0$. Entries depending on $\min(i, j)$ or $|i-j|$: subtract consecutive rows. Vandermonde: $\det[x_i^{j-1}] = \prod_{i<j}(x_j - x_i)$. A determinant with a parameter is a *polynomial* in it: find its roots and its degree.
5. **Finite fields.** Reduction mod p is a ring homomorphism $\mathbb{Z} \to \mathbb{F}_p$, so it commutes with sums, products, and determinants: an integer matrix has $\det \not\equiv 0 \pmod p$ iff it is invertible over $\mathbb{F}_p$. Rank and row reduction work over $\mathbb{F}_p$ as usual; over $\mathbb{F}_2$ signs vanish. $|GL_n(\mathbb{F}_q)| = \prod_{i=0}^{n-1}(q^n - q^i)$ (choose the rows one at a time, each outside the span of the previous ones).

Example 1 (Rank: a one-sided inverse is two-sided)**.** If A, B are $n \times n$ and $AB = I$, then $BA = I$. *If $Bx = 0$ then $x = ABx = 0$, so $\ker B = 0$ and by rank–nullity $\operatorname{rank} B = n$: B is invertible. Then $A = A(BB^{-1}) = (AB)B^{-1} = B^{-1}$, so $BA = B^{-1}B = I$. Finite dimension is essential: on the space of sequences, the left shift A and right shift B satisfy $AB = I \ne BA$. This is how rank enters a problem that never mentions it.*

Example 2 (A determinant is a polynomial: Vandermonde 3×3)**.** Compute $V = \det\begin{pmatrix}1 & 1 & 1\\ a & b & c\\ a^2 & b^2 & c^2\end{pmatrix}$.

Regard V as a polynomial in c of degree at most 2. It vanishes at $c = a$ and at $c = b$, because then two columns coincide. Hence $V = K\,(c-a)(c-b)$ where K is the coefficient of c^2; expanding along the third column, that coefficient is the 2×2 minor $\det\begin{pmatrix}1 & 1\\ a & b\end{pmatrix} = b - a$. So $V = (b-a)(c-a)(c-b)$. Induction gives $\det[x_i^{j-1}]_{i,j=1}^n = \prod_{i<j}(x_j - x_i)$: roots, degree and one leading coefficient determine a polynomial.

Example 3 (Eigenvalues: a matrix equation forces a sign)**.** If A is a real $n \times n$ matrix with $A^3 = A + I$, then $\det A > 0$. *Every eigenvalue $\lambda \in \mathbb{C}$ of A satisfies $\lambda^3 - \lambda - 1 = 0$ (apply $A^3 - A - I$ to an eigenvector). Let $f(t) = t^3 - t - 1$. For $t \le 0$ we have $f(t) < 0$: if $t \le -1$ then $t(t^2 - 1) \le 0$, and if $-1 < t \le 0$ then $t(t^2 - 1) = |t|(1 - t^2) < 1$. So f has no root in $(-\infty, 0]$: every real eigenvalue of A is positive, and since the characteristic polynomial of A is real, the nonreal eigenvalues come in conjugate pairs $\mu, \bar{\mu}$, each pair contributing $|\mu|^2 > 0$. Hence* $\det A$*, the product of the eigenvalues with multiplicity, is positive.*

Pitfall. (1) "$\det A = \prod \lambda_i$" and "$\operatorname{tr} A = \sum \lambda_i$" are statements about *complex* eigenvalues with *algebraic* multiplicity; do not assume real or distinct eigenvalues, and do not assume diagonalizability (if you need it, say why: symmetric, or n independent eigenvectors in hand). (2) $\operatorname{tr}(ABC) = \operatorname{tr}(BCA)$ by cyclic shift, but $\operatorname{tr}(ABC) \neq \operatorname{tr}(BAC)$ in general. (3) $\operatorname{rank}(AB)$ and $\operatorname{rank}(BA)$ can differ, and $AB = 0$ does not give $A = 0$ or $B = 0$. (4) Over $\mathbb{F}_p$ the argument "$\operatorname{tr} I = n \neq 0$" fails when $p \mid n$.

How to recognize this technique on the exam. A Putnam problem that mentions a matrix is nearly always asking you to choose the right scalar invariant. "Show no such matrices exist" or "n must be even": take det or tr of both sides. "Compute det of a matrix defined by a formula": look for rank-one structure, row differences, block structure, or a polynomial in a parameter, and always compute $n = 1, 2, 3$ first to guess the answer. "1 is an eigenvalue," "$A^k = I$," "$A^2 = A$": the eigenvalues satisfy the same polynomial equation, and conjugate pairs give parity information. "Odd determinant," "entries 0 or 1," "integer matrix": reduce mod 2 and do linear algebra over $\mathbb{F}_2$. Rank is in play whenever a problem says "linearly independent" or "injective."

Suggested timing (15 minutes). 0–4: toolbox items 1 and 3, Example 1 (rank forces the two-sided inverse). 4–8: eigenvalues and trace, Example 3, the conjugate-pair parity trick. 8–11: structured determinants, Example 2, "compute $n = 1, 2, 3$ first." 11–14: reduction mod p and the $\mathbb{F}_2$ mindset (signs disappear, det odd means invertible). 14–15: pitfall (1); then groups start on Problems 1–4.