Week 7: Linear Algebra: Determinants, Eigenvalues, Rank, Structured Matrices

Handout · Thu Nov 12 / Fri Nov 13

Turn a matrix statement into a scalar statement. Take det or tr of both sides of a matrix identity: $\det(AB) = \det A \det B$, $\det A = \prod \lambda_i$, $\operatorname{tr} A = \sum \lambda_i$, $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, and every eigenvalue of A satisfies any polynomial equation that A does; for real matrices, nonreal eigenvalues come in conjugate pairs. **Rank:** $\operatorname{rank}(AB) \le \min(\operatorname{rank} A, \operatorname{rank} B)$, $\operatorname{rank} A + \dim\ker A = n$; a rank-$r$ matrix $M - \lambda I$ gives λ multiplicity $\ge n-r$. **Structured determinants:** row differences, rank-one pieces uv^T, the all-ones matrix J, block forms, and "det is a polynomial in the parameter"; always compute $n = 1, 2, 3$ first. **Finite fields:** reduction mod p commutes with det, so "odd determinant" means "invertible over $\mathbb{F}_2$," where signs disappear.

Problem 1. ⋆ *(warm-up)* Let $n \ge 1$.

(a) Let A be a real $n \times n$ matrix with $A^2 = -I$. Show that n is even.

(b) Show that there are no real $n \times n$ matrices A, B with $AB - BA = I$.

Problem 2. ⋆ *(warm-up)* Let $n \ge 2$ and let a, b be real numbers.

(a) Compute the determinant of the $n \times n$ matrix whose diagonal entries all equal a and whose off-diagonal entries all equal b.

(b) Deduce that the $n \times n$ matrix with 0 on the diagonal and 1 everywhere else is invertible over $\mathbb{F}_2$ if and only if n is even.

Problem 3. ⋆⋆ *(Putnam 2014 A2)* Let A be the $n \times n$ matrix whose entry in the i-th row and j-th column is

$$\frac{1}{\min(i,j)}$$

for $1 \le i, j \le n$. Compute $\det(A)$.

Problem 4. ⋆⋆ *(Putnam 2019 B3)* Let Q be an n-by-n real orthogonal matrix, and let $u \in \mathbb{R}^n$ be a unit column vector (that is, $u^T u = 1$). Let $P = I - 2uu^T$, where I is the n-by-n identity matrix. Show that if 1 is not an eigenvalue of Q, then 1 is an eigenvalue of PQ.

Problem 5. ⋆⋆⋆ *(Putnam 2018 A2)* Let $S_1, S_2, \ldots, S_{2^n-1}$ be the nonempty subsets of $\{1, 2, \ldots, n\}$ in some order, and let M be the $(2^n - 1) \times (2^n - 1)$ matrix whose (i, j) entry is

$$m_{ij} = \begin{cases} 0 & \text{if } S_i \cap S_j = \emptyset, \\ 1 & \text{otherwise.} \end{cases}$$

Calculate the determinant of M.

Problem 6. ⋆⋆⋆⋆ *(Putnam 2021 B5)* Say that an n-by-n matrix $A = (a_{ij})_{1 \le i,j \le n}$ with integer entries is *very odd* if, for every nonempty subset S of $\{1, 2, \ldots, n\}$, the $|S|$-by-$|S|$ submatrix $(a_{ij})_{i,j \in S}$ has odd determinant. Prove that if A is very odd, then A^k is very odd for every $k \ge 1$.

Problems 1–4 are the in-session targets; 5–6 are take-home. Solutions will be posted after Friday's session.