Week 6: Inequalities and Analysis

Solutions · Thu Nov 5 / Fri Nov 6

Problem 1. ⋆ *(warm-up) For $a, b, c > 0$: (a) $(a+b)(b+c)(c+a) \geq 8abc$; (b) $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$.*
Key idea. Equality is at $a = b = c$; apply AM–GM factor by factor in (a), and Cauchy–Schwarz in Engel form after writing each fraction with a square upstairs in (b).
Solution. (a) By AM–GM on two terms, $a + b \geq 2\sqrt{ab}$, $b + c \geq 2\sqrt{bc}$, $c + a \geq 2\sqrt{ca}$. All three sides are positive, so multiplying gives $(a+b)(b+c)(c+a) \geq 8\sqrt{ab \cdot bc \cdot ca} = 8abc$, with equality iff $a = b = c$.

(b) Write $\frac{a}{b+c} = \frac{a^2}{a(b+c)}$ and similarly for the other terms. The denominators $a(b+c)$, $b(c+a)$, $c(a+b)$ are positive, so the Engel form of Cauchy–Schwarz gives

$$\sum \frac{a^2}{a(b+c)} \;\geq\; \frac{(a+b+c)^2}{a(b+c)+b(c+a)+c(a+b)} = \frac{(a+b+c)^2}{2(ab+bc+ca)}.$$

It remains to show $(a+b+c)^2 \geq 3(ab+bc+ca)$. Expanding, this is $a^2+b^2+c^2 \geq ab+bc+ca$, which is half of $(a-b)^2+(b-c)^2+(c-a)^2 \geq 0$. Hence the sum is at least $\frac{3}{2}$, with equality iff $a = b = c$. □

Remark. An AM–GM route for (b): adding 1 to each fraction, the claim becomes $(a+b+c)\big(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\big) \geq \frac{9}{2}$. With $x = b+c$, $y = c+a$, $z = a+b$ this is $(x+y+z)(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}) \geq 9$, which is AM–GM (or Cauchy–Schwarz) applied to x, y, z and to $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$.

Problem 2. ⋆ *(classical) f continuous on $[0,1]$, differentiable on $(0,1)$, $f(0) = 0$, $f(1) = 1$. Show there are distinct $a, b \in (0,1)$ with $1/f'(a) + 1/f'(b) = 2$.*
Key idea. Split $[0,1]$ at a point where $f = \frac{1}{2}$ and apply the mean value theorem on each piece; the reciprocals of the resulting slopes are the lengths of the pieces.
Solution. Since f is continuous with $f(0) = 0 < \frac{1}{2} < 1 = f(1)$, the intermediate value theorem gives $c \in (0,1)$ with $f(c) = \frac{1}{2}$. Now apply the mean value theorem on $[0,c]$ and on $[c,1]$ (on each, f is continuous on the closed interval and differentiable inside): there are $a \in (0,c)$ and $b \in (c,1)$ with

$$f'(a) = \frac{f(c)-f(0)}{c-0} = \frac{1}{2c}, \qquad f'(b) = \frac{f(1)-f(c)}{1-c} = \frac{1}{2(1-c)}.$$

Both values are positive, so their reciprocals make sense: $\frac{1}{f'(a)} + \frac{1}{f'(b)} = 2c + 2(1-c) = 2$. Finally $a < c < b$, so $a \neq b$. □

Remark. The same argument with points $c_1 < \cdots < c_{n-1}$, chosen successively by the IVT on $[c_{k-1}, 1]$, where $f(c_k) = k/n$ produces distinct $a_1, \ldots, a_n$ with $\sum 1/f'(a_k) = n$. The lesson is that the MVT gives you a point on each interval you choose, and the choice of intervals is where the problem is solved.

Problem 3. ⋆⋆ *(Putnam 2003 A2) For nonnegative a_i, b_i: $(a_1 \cdots a_n)^{1/n} + (b_1 \cdots b_n)^{1/n} \leq [(a_1 + b_1) \cdots (a_n + b_n)]^{1/n}$.*
Key idea. Divide through by the right-hand side so that both terms become geometric means of numbers x_i, y_i with $x_i + y_i = 1$, then apply AM–GM to each.
Solution. If $a_i + b_i = 0$ for some i, then $a_i = b_i = 0$ and both sides equal 0. So assume $a_i + b_i > 0$ for every i, and put

$$x_i = \frac{a_i}{a_i + b_i}, \qquad y_i = \frac{b_i}{a_i + b_i}, \qquad \text{so that } x_i, y_i \geq 0 \text{ and } x_i + y_i = 1.$$

Dividing the desired inequality by the positive number $[(a_1+b_1)\cdots(a_n+b_n)]^{1/n}$, it becomes

$$(x_1x_2\cdots x_n)^{1/n}+(y_1y_2\cdots y_n)^{1/n}\le 1.$$

Now AM–GM applies to the nonnegative numbers $x_1,\dots,x_n$ and separately to $y_1,\dots,y_n$:

$$(x_1\cdots x_n)^{1/n}+(y_1\cdots y_n)^{1/n}\le\frac{x_1+\cdots+x_n}{n}+\frac{y_1+\cdots+y_n}{n}=\frac{1}{n}\sum_{i=1}^{n}(x_i+y_i)=1.$$

Equality holds iff all x_i are equal (and hence all y_i are equal), i.e. iff one of (a_i), (b_i) is a nonnegative multiple of the other, or some $a_i=b_i=0$. □

Remark. Convexity gives the same result: if all $a_i,b_i>0$, set $s_i=\log(b_i/a_i)$ and divide by $(a_1\cdots a_n)^{1/n}$; the inequality becomes $1+\exp(\frac{1}{n}\sum s_i)\le\prod(1+e^{s_i})^{1/n}$, and taking logarithms this is Jensen's inequality for the convex function $\varphi(s)=\log(1+e^s)$, whose second derivative is $e^s/(1+e^s)^2>0$. The statement says the geometric mean is *superadditive*, equivalently concave, on $[0,\infty)^n$.

Problem 4. ⋆⋆ *(Putnam 2014 B2)* $-1\le f\le 1$ *on* $[1,3]$ *and* $\int_1^3 f=0$. *Maximize* $\int_1^3 f(x)/x\,dx$.
Key idea. Subtract a multiple of the constraint so the weight $\frac{1}{x}-\lambda$ changes sign exactly where the optimal f (which takes only the values ±1) switches from $+1$ to -1; the constraint forces the switch at $x=2$, so $\lambda=\frac12$.

Solution. The answer is $\log\frac43$. We assume f is integrable, as the problem intends.

The bound. Since $\int_1^3 f=0$, for every constant λ we have

$$\int_1^3\frac{f(x)}{x}\,dx=\int_1^3 f(x)\Big(\frac1x-\lambda\Big)dx.$$

Take $\lambda=\frac12$, so that $w(x)=\frac1x-\frac12$ is ≥0 on $[1,2]$ and ≤0 on $[2,3]$. Pointwise, $f(x)w(x)\le|w(x)|$: on $[1,2]$ because $f\le1$ and $w\ge0$, on $[2,3]$ because $f\ge-1$ and $w\le0$. Integrating,

$$\int_1^3\frac{f(x)}{x}\,dx\le\int_1^3\Big|\frac1x-\frac12\Big|dx=\int_1^2\Big(\frac1x-\frac12\Big)dx+\int_2^3\Big(\frac12-\frac1x\Big)dx=\Big(\log2-\frac12\Big)+\Big(\frac12-\log\frac32\Big)=\log\frac43.$$

The example. Let $g=1$ on $[1,2]$ and $g=-1$ on $(2,3]$. Then $-1\le g\le1$, $\int_1^3 g=1-1=0$, and $\int_1^3 g(x)/x\,dx=\log2-\log\frac32=\log\frac43$. So the maximum is $\log\frac43$. □

Remark. Why $\lambda=\frac12$: the optimizer should put $f=+1$ where the weight $1/x$ is largest and $f=-1$ where it is smallest, and the constraint $\int f=0$ forces the two regions to have equal length, so the switch is at $x=2$ and λ must equal the weight there, $\frac12$. The same argument handles any decreasing weight and any constraint $\int f=c$.

Problem 5. ⋆⋆⋆ *(Putnam 2012 B4)* $a_0=1$, $a_{n+1}=a_n+e^{-a_n}$. *Does* $a_n-\log n$ *have a finite limit?*
Key idea. Yes: a_n tracks the solution $y=\log x$ of $y'=e^{-y}$; prove $a_n>\log(n+1)$ by induction, then show $a_n-\log n$ is decreasing and bounded below.

Solution. We use two standard estimates, both from $\log(1+t)=\int_0^t\frac{ds}{1+s}$ with $t>0$: since $\frac{1}{1+t}<\frac{1}{1+s}<1$ on $(0,t)$,

$$\frac{t}{1+t}<\log(1+t)<t\qquad(t>0).\tag{$*$}$$

Step 1: $a_n > \log(n+1)$ for all $n \geq 0$. True for $n = 0$ since $a_0 = 1 > 0$. The function $\varphi(u) = u + e^{-u}$ has $\varphi'(u) = 1 - e^{-u} > 0$ for $u > 0$, so φ is increasing on $[0, \infty)$. If $a_n > \log(n+1) \geq 0$, then

$$a_{n+1} = \varphi(a_n) > \varphi(\log(n+1)) = \log(n+1) + \frac{1}{n+1} > \log(n+1) + \log\Big(1 + \frac{1}{n+1}\Big) = \log(n+2),$$

using the right half of $(*)$ with $t = \frac{1}{n+1}$. This completes the induction.

Step 2: $d_n = a_n - \log n$ is decreasing for $n \geq 1$. Using Step 1 (so $e^{-a_n} < \frac{1}{n+1}$) and then the left half of $(*)$ with $t = \frac{1}{n}$ (so $\frac{t}{1+t} = \frac{1}{n+1}$),

$$d_{n+1} - d_n = e^{-a_n} - \log\frac{n+1}{n} < \frac{1}{n+1} - \log\Big(1 + \frac{1}{n}\Big) < 0.$$

Step 3: d_n is bounded below. By Step 1, $d_n > \log(n+1) - \log n > 0$.

A decreasing sequence bounded below converges, so $a_n - \log n$ has a finite limit (and it is ≥ 0). □

Remark. The limit is in fact 0, and the substitution $b_n = e^{a_n}$ shows why: the recurrence becomes $b_{n+1} = b_n e^{1/b_n}$. Since $e^t - 1 - t \leq \frac{e}{2}t^2$ for $0 \leq t \leq 1$ (since $e^t - 1 - t = \sum_{k\geq 2} t^k/k! \leq t^2 \sum_{k\geq 2} 1/k! = (e-2)t^2$) and $b_n \geq n + e \geq 1$ by induction, we get $0 \leq b_{n+1} - b_n - 1 = b_n\big(e^{1/b_n} - 1 - \frac{1}{b_n}\big) \leq \frac{e}{2b_n} \leq \frac{e}{2(n+1)}$. Summing, $n + e \leq b_n \leq n + e + \frac{e}{2}(1 + \frac{1}{2} + \cdots + \frac{1}{n}) = n + O(\log n)$, so $b_n/n \to 1$ and $a_n - \log n = \log(b_n/n) \to 0$. This is the general principle: a recurrence $a_{n+1} - a_n = F(a_n)$ behaves like the differential equation $y' = F(y)$, and a change of variables that makes the equation linear ($y = \log x$ here) makes the recurrence tractable.

Problem 6. ⋆⋆⋆⋆ *(Putnam 2023 A3) Find the smallest $r > 0$ for which there are differentiable $f, g\colon \mathbb{R} \to \mathbb{R}$ with $f(0) > 0$, $g(0) = 0$, $|f'| \leq |g|$, $|g'| \leq |f|$, and $f(r) = 0$.*

Key idea. The pair $(\cos, \sin)$ shows $r = \pi/2$ is possible; for the lower bound, control two derived quantities: $f^2 + g^2$ cannot vanish (Grönwall), and the angle $\arctan(g/f)$ moves at speed at most 1, yet must travel from 0 to $\pm\pi/2$ before f can vanish.

Solution. The answer is $r = \pi/2$.

Construction. Take $f(x) = \cos x$, $g(x) = \sin x$. Then $f(0) = 1 > 0$, $g(0) = 0$, $|f'| = |\sin x| = |g|$, $|g'| = |\cos x| = |f|$, and $f(\pi/2) = 0$. So the smallest r is at most $\pi/2$.

Lower bound. Suppose f, g satisfy (a)–(e) for some $r > 0$. Let $r_0 = \inf\{x > 0 : f(x) = 0\}$; the set is nonempty (it contains r) and f is continuous, so $f(r_0) = 0$, and $r_0 > 0$ because $f(0) > 0$ keeps f positive on a neighborhood of 0. Moreover $f > 0$ on $[0, r_0)$: it is positive at 0, has no zero on $(0, r_0)$, and is continuous. We show $r_0 \geq \pi/2$; since $r \geq r_0$, this proves the claim.

Step 1: $k = f^2 + g^2$ never vanishes. k is differentiable with $k' = 2ff' + 2gg'$, so by (c), (d) and $2|f||g| \leq f^2 + g^2$,

$$|k'| \leq 2|f||g| + 2|g||f| \leq 2(f^2 + g^2) = 2k.$$

Hence $(e^{2x}k)' = e^{2x}(k' + 2k) \geq 0$, so $e^{2x}k(x)$ is nondecreasing (mean value theorem) and $k(x) \geq e^{-2x}k(0) = e^{-2x}f(0)^2 > 0$ for all $x \geq 0$. In particular $k(r_0) = g(r_0)^2 > 0$, so $g(r_0) \neq 0$.

Step 2: the angle moves slowly. On $[0, r_0)$ we have $f > 0$, so $h(x) = \arctan\big(g(x)/f(x)\big)$ is differentiable there, with

$$h' = \frac{g'f - gf'}{f^2 + g^2}, \qquad |h'| \leq \frac{|g'||f| + |g||f'|}{f^2 + g^2} \leq \frac{|f|^2 + |g|^2}{f^2 + g^2} = 1$$

by (c) and (d). Since $h(0) = \arctan 0 = 0$, the mean value theorem gives $|h(x)| = |h(x) - h(0)| \le x$ for all $x \in [0, r_0)$.

Step 3: conclusion. As $x \to r_0^-$, $f(x) \to f(r_0) = 0$ with $f(x) > 0$, and $g(x) \to g(r_0) \neq 0$, so $g(x)/f(x) \to \pm\infty$ and $|h(x)| \to \pi/2$. Letting $x \to r_0^-$ in $|h(x)| \le x$ yields $\pi/2 \le r_0$. Thus every admissible r satisfies $r \ge r_0 \ge \pi/2$, and the smallest such r is $\pi/2$. □

Remark. The two auxiliary functions are exactly the polar coordinates of the curve (f, g): Step 1 says the radius cannot collapse faster than e^{-x}, and Step 2 says the angular speed is at most 1. The hypotheses (c) and (d) are what one gets from $f' = -g$, $g' = f$ up to sign, so it is natural to compare with $(\cos, \sin)$, whose angle moves at speed exactly 1.