Week 6: Inequalities and Analysis

Lecture notes · Thu Nov 5 / Fri Nov 6

Toolbox: six moves, and one habit.

1. **AM–GM.** For $x_i \ge 0$: $\frac{x_1+\cdots+x_n}{n} \ge (x_1 \cdots x_n)^{1/n}$, equality iff all x_i are equal; weighted form $\sum w_i x_i \ge \prod x_i^{w_i}$ for $w_i > 0$, $\sum w_i = 1$. Most applications first rescale the x_i so the equality case becomes "all equal."
2. **Cauchy–Schwarz.** $(\sum a_i b_i)^2 \le \sum a_i^2 \sum b_i^2$, also $(\int fg)^2 \le \int f^2 \int g^2$. Engel form: for $b_i > 0$, $\sum a_i^2/b_i \ge (\sum a_i)^2/\sum b_i$; use it when a sum of fractions has squares upstairs, or can be made to.
3. **Convexity.** If $\varphi'' \ge 0$ on an interval containing the x_i, Jensen gives $\sum w_i \varphi(x_i) \ge \varphi(\sum w_i x_i)$ (reverse for concave). When φ is not convex on the whole range, the *tangent line trick* (prove $\varphi(x) \ge$ its tangent at the equality point by hand) often still works.
4. **Integrals.** Bound integrands pointwise, splitting the domain where a sign changes. Absorb a constraint $\int f = c$ by writing $\int fw = \int f\,(w-\lambda) + \lambda c$, with λ chosen so $w - \lambda$ changes sign where the optimizer switches.
5. **Sequences.** Monotone and bounded $\Rightarrow$ convergent, and then the limit satisfies the fixed-point equation. Telescope: look for $x_n = F(x_n) - F(x_{n+1})$. Substitute ($b_n = e^{a_n}$, $b_n = 1/a_n$) to turn a recurrence into "$b_{n+1} \approx b_n + 1$."
6. **Rolle and MVT.** $f(b) - f(a) = f'(c)(b-a)$ for some $c \in (a,b)$. Between consecutive zeros of f lies a zero of f', so k zeros of f give $k-1$ of f'. To handle $f + \lambda^{-1} f'$, note $(e^{\lambda x} f)' = \lambda e^{\lambda x}(f + \lambda^{-1} f')$. Grönwall: if $|k'| \le Ck$ with $k \ge 0$, then $(e^{Cx}k)' \ge 0$, so $k(x) \ge k(0)e^{-Cx}$.

The habit: before proving any bound, guess where equality holds (all variables equal, a boundary point, a bang-bang function $f = \pm 1$). The guess tells you which inequality to apply and how to weight it, and a "how large can it be" answer is incomplete without the example.

Example 1 (Telescoping and limits, Putnam 2016 B1)**.** Let $x_0 = 1$ and $x_{n+1} = \ln(e^{x_n} - x_n)$. Show $\sum_{n\ge 0} x_n$ converges and find its sum. *Exponentiate the recurrence:* $e^{x_{n+1}} = e^{x_n} - x_n$*, that is,* $x_n = e^{x_n} - e^{x_{n+1}}$*, a telescoping shape. First check the sequence makes sense:* $u \mapsto e^u - u$ *is increasing on* $u > 0$ *with value* 1 *at* $u = 0$*, so by induction* $e^{x_n} - x_n > 1$ *and* $x_n > 0$ *for all* n*. Then* $e^{x_{n+1}} = e^{x_n} - x_n < e^{x_n}$*, so* (x_n) *is decreasing and bounded below by* 0*, hence converges to some* $L \ge 0$*. Taking limits in* $x_n = e^{x_n} - e^{x_{n+1}}$ *gives* $L = e^L - e^L = 0$*. Now* $x_0 + \cdots + x_n = e^{x_0} - e^{x_{n+1}} \to e^1 - e^0 = e - 1$*. The moves: linearize by exponentiating, prove convergence* before *taking limits, then telescope.*

Example 2 (Rolle with an exponential multiplier, Putnam 2015 B1)**.** If f is three times differentiable on $\mathbb{R}$ with at least five distinct real zeros, then $f + 6f' + 12f'' + 8f'''$ has at least two distinct real zeros. *The coefficients are* $(1+2D)^3 = 1 + 6D + 12D^2 + 8D^3$ *applied to* f*, where* $D = d/dx$*. Since* $(e^{x/2}f)' = \frac{1}{2}e^{x/2}(f + 2f')$*, three applications give* $(e^{x/2}f)''' = \frac{1}{8}e^{x/2}(f + 6f' + 12f'' + 8f''')$*. Let* $g = e^{x/2}f$*, which has the same five zeros as* f*. Rolle's theorem between consecutive zeros gives* g' *at least four distinct zeros,* g'' *at least three, and* g''' *at least two. As* $e^{x/2} \ne 0$*, those are zeros of* $f + 6f' + 12f'' + 8f'''$*. Whenever you see* $f + cf'$*, reach for* $e^{x/c}$*.*

Example 3 (Tangent line trick)**.** For $a, b, c > 0$ with $a + b + c = 3$, show $\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} \ge \frac{3}{2}$. *Equality should be at* $a = b = c = 1$*, but* $\varphi(x) = 1/(x^2+1)$ *has* $\varphi''(x) = (6x^2-2)/(x^2+1)^3$*,* *negative for $x < 1/\sqrt{3}$, so φ is not convex on $(0, 3)$ and Jensen does not apply. Instead compare with the tangent line at $x = 1$, which is $y = 1 - \frac{x}{2}$: the claim $\frac{1}{x^2+1} \geq 1 - \frac{x}{2}$ is equivalent (multiply by $2(x^2 + 1) > 0$) to $2 \geq (2 - x)(x^2 + 1)$, i.e. $x^3 - 2x^2 + x = x(x - 1)^2 \geq 0$, true for $x \geq 0$. Summing the three tangent-line bounds gives $3 - \frac{a+b+c}{2} = \frac{3}{2}$. Write the inequality with the equality point in view, then check it with a factorization.*

Pitfall. (1) AM–GM needs nonnegative terms and Engel needs positive denominators; AM–GM applied to $a - b$ and c proves nothing. (2) Jensen needs convexity on an interval containing *all* the points, in the direction you want; check φ'' on the whole range. (3) "Let $L = \lim a_n$ and substitute into the recurrence" gives a candidate for the limit, not a proof that it exists; monotonicity or a contraction estimate must come first. (4) The MVT hands you a point c you cannot choose, and different applications give different c's. (5) For "how large can X be," a bound with no example, or an example with no bound, scores at most 1 or 2.

How to recognize this technique on the exam. A symmetric expression to be bounded, especially with a constraint like $\sum x_i = 1$, calls for AM–GM, Cauchy–Schwarz, or Jensen; find the equality case first and choose the tool that has equality there. "How large can $\int fw$ be" with f bounded and one integral constraint is bang-bang: the optimizer is ± 1 with a switch where a shifted weight changes sign. A recurrence with a question about $a_n - \log n$ or $\sqrt{n}\, a_n$ is asking for asymptotics; compare with the differential equation $y' = F(y)$ and substitute to make the recurrence nearly linear. Hypotheses such as "differentiable," "$f(0) = f(1) = 0$," or a bound on $|f'|$ in terms of $|f|$ or $|g|$ mean Rolle, the MVT, or Grönwall; count zeros, or build a quantity ($f^2 + g^2$, $\arctan(g/f)$, $e^{\lambda x} f$) whose derivative you can control.

Suggested timing (15 minutes). 0–4: AM–GM, Cauchy–Schwarz (Engel), and "guess the equality case"; state Jensen and do Example 3 quickly. 4–8: integrals as pointwise comparison, and the λ trick for a constraint (preview of Problem 4 without solving it). 8–12: Example 1, stressing convergence before limits and telescoping. 12–15: Example 2 (Rolle, exponential multiplier), one sentence on Grönwall; then groups start on Problems 1–4.