Week 6: Inequalities and Analysis

Handout · Thu Nov 5 / Fri Nov 6

Inequalities: AM–GM ($\frac{x_1+\cdots+x_n}{n} \geq (x_1\cdots x_n)^{1/n}$ for $x_i \geq 0$, equality iff all equal), Cauchy–Schwarz in Engel form ($\sum a_i^2/b_i \geq (\sum a_i)^2/\sum b_i$ for $b_i > 0$), and Jensen (φ convex $\Rightarrow$ $\frac{1}{n}\sum \varphi(x_i) \geq \varphi(\frac{1}{n}\sum x_i)$); when Jensen fails, try the tangent line at the equality point. Always guess the equality case first. **Integrals:** compare integrands pointwise; absorb a constraint $\int f = c$ by subtracting a constant λ from the weight. **Sequences:** monotone and bounded $\Rightarrow$ convergent; telescope; substitute ($b_n = e^{a_n}$) to linearize a recurrence. **Derivatives:** Rolle and the mean value theorem produce points you cannot choose but can count; $e^{\lambda x} f$ turns $f + \lambda^{-1} f'$ into a derivative; $|k'| \leq Ck$ with $k \geq 0$ forces $k(x) \geq k(0)e^{-Cx}$ for $x \geq 0$ (Grönwall).

Problem 1. ⋆ *(warm-up)* Let a, b, c be positive real numbers.

(a) Show that $(a+b)(b+c)(c+a) \geq 8abc$.

(b) (Nesbitt) Show that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \;\geq\; \frac{3}{2}.$$

Problem 2. ⋆ *(classical)* Let $f\colon [0,1] \to \mathbb{R}$ be continuous on $[0,1]$ and differentiable on $(0,1)$, with $f(0) = 0$ and $f(1) = 1$. Prove that there exist distinct points $a, b \in (0,1)$ such that

$$\frac{1}{f'(a)} + \frac{1}{f'(b)} = 2.$$

Problem 3. ⋆⋆ *(Putnam 2003 A2)* Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be nonnegative real numbers. Show that

$$(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \;\leq\; \big[(a_1+b_1)(a_2+b_2)\cdots(a_n+b_n)\big]^{1/n}.$$

Problem 4. ⋆⋆ *(Putnam 2014 B2)* Suppose that f is a function on the interval $[1,3]$ such that $-1 \leq f(x) \leq 1$ for all x and $\int_1^3 f(x)\,dx = 0$. How large can $\displaystyle\int_1^3 \frac{f(x)}{x}\,dx$ be?

Problem 5. ⋆⋆⋆ *(Putnam 2012 B4)* Suppose that $a_0 = 1$ and that $a_{n+1} = a_n + e^{-a_n}$ for $n = 0, 1, 2, \ldots$. Does $a_n - \log n$ have a finite limit as $n \to \infty$? (Here $\log n = \log_e n = \ln n$.)

Problem 6. ⋆⋆⋆⋆ *(Putnam 2023 A3)* Determine the smallest positive real number r such that there exist differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ satisfying

(a) $f(0) > 0$,

(b) $g(0) = 0$,

(c) $|f'(x)| \leq |g(x)|$ for all x,

(d) $|g'(x)| \leq |f(x)|$ for all x, and

(e) $f(r) = 0$.

Problems 1–4 are the in-session targets; 5–6 are take-home. Solutions will be posted after Friday's session.