Week 5: Combinatorics: Bijections, Double Counting, Recursions, Generating Functions

Solutions · Thu Oct 29 / Fri Oct 30

Problem 1. ⋆ *(warm-up) a_n counts subsets of $[n] = \{1, \dots, n\}$ with no two consecutive elements. (a) $a_n = F_{n+2}$. (b) Exactly $\binom{n-k+1}{k}$ of them have k elements, so $F_{n+2} = \sum_k \binom{n-k+1}{k}$.*
Key idea. Condition on whether n is in the set; for (b), compress a k-subset with gaps into an arbitrary k-subset of a shorter interval.

Solution. (a) Let $S \subseteq [n]$ have no two consecutive elements, $n \geq 2$. If $n \notin S$, then S is such a subset of $[n-1]$, and every such subset arises; there are a_{n-1} of these. If $n \in S$, then $n-1 \notin S$, so $S \setminus \{n\}$ is such a subset of $[n-2]$, and adjoining n to any such subset of $[n-2]$ gives a valid S; there are a_{n-2} of these. Hence $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. Directly, $a_0 = 1$ (only $\varnothing$) and $a_1 = 2$, while $F_2 = 1$ and $F_3 = 2$. Since (a_n) and (F_{n+2}) satisfy the same recurrence with the same two initial values, $a_n = F_{n+2}$ for all $n \geq 0$ by induction.

(b) Let $S = \{s_1 < s_2 < \cdots < s_k\}$ be a k-subset of $[n]$. Define $\varphi(S) = \{s_1,\ s_2-1,\ s_3-2,\ \dots,\ s_k-(k-1)\}$. The condition "no two consecutive" means $s_{i+1} \geq s_i + 2$ for all i, i.e. $s_{i+1} - i > s_i - (i-1)$: exactly the condition that the listed numbers are strictly increasing, so that $\varphi(S)$ is a k-subset of $[n-k+1]$ (its largest element is $s_k - (k-1) \leq n-k+1$, its smallest is $s_1 \geq 1$). Conversely, given $t_1 < \cdots < t_k$ in $[n-k+1]$, the set $\{t_1, t_2+1, \dots, t_k+(k-1)\}$ lies in $[n]$, has gaps of at least 2, and is sent to $\{t_i\}$ by φ. So φ is a bijection onto the k-subsets of $[n-k+1]$, of which there are $\binom{n-k+1}{k}$. Summing over k and using (a), $F_{n+2} = a_n = \sum_{k \geq 0} \binom{n-k+1}{k}$. □

Problem 2. ⋆ *(Mantel, 1907) A triangle-free graph on n vertices has at most $n^2/4$ edges.*
Key idea. For an edge uv the neighborhoods of u and v are disjoint, so $\deg u + \deg v \leq n$; summing over edges double-counts $\sum_v (\deg v)^2$.

Solution. Write $d(v)$ for the degree of v and e for the number of edges. If uv is an edge, no vertex is adjacent to both u and v (that would be a triangle), so the neighborhoods of u and v are disjoint subsets of the vertex set and $d(u) + d(v) \leq n$. Summing over all edges,

$$\sum_{uv \in E} \big(d(u) + d(v)\big) \leq ne.$$

Now double count the left side: the term $d(v)$ appears once for each edge at v, i.e. $d(v)$ times, so the sum equals $\sum_v d(v)^2$. By the Cauchy–Schwarz inequality (equivalently, the mean of squares is at least the square of the mean), $\sum_v d(v)^2 \geq \frac{1}{n}\big(\sum_v d(v)\big)^2 = \frac{(2e)^2}{n}$, where $\sum_v d(v) = 2e$ is the handshake lemma, itself a double count of (vertex, incident edge) pairs. Combining, $4e^2/n \leq ne$, so (dividing by $e > 0$; the case $e = 0$ is trivial) $e \leq n^2/4$. □

Remark. The bound is attained by the complete bipartite graph with parts of sizes $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$. A second proof uses the extremal principle: pick v of maximum degree Δ; its neighbors are pairwise non-adjacent, so every edge has an endpoint in $V \setminus N(v)$, a set of $n - \Delta$ vertices (including v itself), each of degree at most Δ, giving $e \leq (n-\Delta)\Delta \leq n^2/4$.

Problem 3. ⋆⋆ *(Putnam 2003 A1) Count the ways to write $n = a_1 + \cdots + a_k$ with $k \geq 1$ arbitrary and $a_1 \leq \cdots \leq a_k \leq a_1 + 1$.*
Key idea. The number of parts k determines the representation completely, and every $k \in \{1, \dots, n\}$ occurs: the representations are in bijection with $\{1, \dots, n\}$.

Solution. The answer is n. Fix k and suppose $n = a_1 + \cdots + a_k$ is a representation with k parts. Every part lies in $\{a_1, a_1 + 1\}$; let r be the number of parts equal to $a_1 + 1$. Since a_1 itself is a part, $0 \le r \le k - 1$, and $n = ka_1 + r$. By the uniqueness of division with remainder, $a_1 = \lfloor n/k \rfloor$ and r is the remainder of n modulo k; and since the parts are listed in nondecreasing order, the representation is the sequence of $k - r$ copies of a_1 followed by r copies of $a_1 + 1$. So there is at most one representation with k parts. Conversely, for $1 \le k \le n$ write $n = kq + r$ with $0 \le r \le k - 1$; then $q = \lfloor n/k \rfloor \ge 1$, and $k - r$ copies of q followed by r copies of $q + 1$ is a valid representation with k parts (all parts positive, nondecreasing, largest at most smallest plus one). For $k > n$ no representation exists, as k positive parts sum to at least k. Thus "number of parts" is a bijection from the set of representations onto $\{1, 2, \ldots, n\}$, and there are exactly n representations. (Check: $n = 4$ gives 4, $2 + 2$, $1 + 1 + 2$, $1 + 1 + 1 + 1$ with $k = 1, 2, 3, 4$.) □

Problem 4. ⋆⋆ *(Putnam 2005 A2) Count rook tours of the* $3 \times n$ *board* S *(columns* $a = 1, \ldots, n$*, rows* $b = 1, 2, 3$*) that visit every cell once, moving between adjacent cells, starting at* $(1, 1)$ *and ending at* $(n, 1)$*.*

Key idea. Condition on how the tour enters the last column: it either sweeps the last column in one vertical stroke or enters it twice in a hairpin, and each case peels off a column and leaves a tour of the $3 \times (n - 1)$ board.

Solution. The answer is 2^{n-2} for $n \ge 2$ (and 0 for $n = 1$, where a tour would have to start and end at the same cell). In a tour, the two endpoints have one tour-neighbor each and every other cell has exactly two, and a tour-neighbor is always a grid-neighbor. Let A_n and B_n be the numbers of tours of the $3 \times n$ board that start at $(1, 1)$ and end at $(n, 1)$, respectively at $(n, 3)$. By hand, $A_2 = B_2 = 1$: the only tour of the 3×2 board from $(1, 1)$ to $(2, 1)$ is $(1, 1), (1, 2), (1, 3), (2, 3), (2, 2), (2, 1)$, and the only one to $(2, 3)$ is $(1, 1), (2, 1), (2, 2), (1, 2), (1, 3), (2, 3)$. We claim that for $n \ge 3$,

$$A_n = A_{n-1} + B_{n-1} \qquad \text{and} \qquad B_n = A_{n-1} + B_{n-1}.$$

Given this, $A_n = B_n$ for all $n \ge 2$, hence $A_n = 2A_{n-1}$ for $n \ge 3$, and $A_n = 2^{n-2}$.

Tours ending at $(n, 1)$. The tour-neighbor of the endpoint $(n, 1)$ is $(n, 2)$ or $(n - 1, 1)$. If it is $(n, 2)$: the cell $(n, 3)$ is not an endpoint, so both of its grid-neighbors $(n - 1, 3)$ and $(n, 2)$ are its tour-neighbors, and the tour ends $(n - 1, 3), (n, 3), (n, 2), (n, 1)$. Deleting these last three cells leaves a tour of the $3 \times (n - 1)$ board from $(1, 1)$ to $(n - 1, 3)$; conversely appending the three cells to any such tour gives a tour ending at $(n, 1)$. This case contributes B_{n-1}. If it is $(n - 1, 1)$: now $(n, 2)$ and $(n, 3)$ are interior cells. The tour-neighbors of $(n, 3)$ are $(n - 1, 3)$ and $(n, 2)$; those of $(n, 2)$ are $(n, 3)$ and $(n - 1, 2)$, because $(n, 1)$ has already used its single tour-neighbor. So the tour contains the run $(n - 1, 2), (n, 2), (n, 3), (n - 1, 3)$ in one direction or the other. Replacing this run by the single step $(n - 1, 2), (n - 1, 3)$ and deleting the final cell $(n, 1)$ gives a tour of the $3 \times (n - 1)$ board ending at $(n - 1, 1)$. Conversely, in every tour of the $3 \times (n - 1)$ board ending at $(n - 1, 1)$ with $n - 1 \ge 2$, the cell $(n - 1, 3)$ is interior, so its tour-neighbors are its two grid-neighbors $(n - 2, 3)$ and $(n - 1, 2)$; in particular the step $(n - 1, 2), (n - 1, 3)$ occurs, and we can expand it back to the run above and append $(n, 1)$. This case contributes A_{n-1}.

Tours ending at $(n, 3)$. The tour-neighbor of $(n, 3)$ is $(n, 2)$ or $(n - 1, 3)$. If it is $(n, 2)$, the interior cell $(n, 1)$ has tour-neighbors $(n - 1, 1)$ and $(n, 2)$, the tour ends $(n - 1, 1), (n, 1), (n, 2), (n, 3)$, and deleting three cells is a bijection with tours ending at $(n - 1, 1)$: contribution A_{n-1}. If it is $(n - 1, 3)$, the interior cells $(n, 1), (n, 2)$ have tour-neighbors $\{(n - 1, 1), (n, 2)\}$ and $\{(n, 1), (n - 1, 2)\}$, so the tour contains the run $(n - 1, 1), (n, 1), (n, 2), (n - 1, 2)$; contracting it to the step $(n - 1, 1), (n - 1, 2)$ and deleting the final cell $(n, 3)$ gives a tour ending at $(n - 1, 3)$, and every tour of the $3 \times (n - 1)$ board ending at $(n - 1, 3)$ contains that step (the cell $(n - 1, 1)$ is interior, with grid-neighbors $(n - 2, 1)$ and $(n - 1, 2)$), so the contraction is a bijection: contribution B_{n-1}. □

Remark. A direct bijection: for a tour P let $T(P)$ be the set of columns i in which P takes a vertical step from row 1 or row 3 into row 2. One checks that $n \in T(P)$, that $|T(P)|$ is even, and that P can be reconstructed from $T(P)$; the subsets of $[n]$ containing n with an even number of elements number 2^{n-2} (for $n \geq 2$).

Problem 5. $\star\star\star$ *(Putnam 2005 B4) $f(m,n)$ is the number of integer n-tuples with $|x_1|+\cdots+|x_n| \leq m$; show $f(m,n) = f(n,m)$.*

Key idea. Classify by the number k of nonzero coordinates; the positions, the signs, and the absolute values are chosen independently, and the absolute values are counted by a bijection with k-subsets of $[m]$.

Solution. Fix k with $0 \leq k \leq \min(m,n)$ and count the tuples with exactly k nonzero coordinates. Choose the set of nonzero positions: $\binom{n}{k}$ ways. Choose their signs: 2^k ways. Finally choose the absolute values $y_1, \ldots, y_k$ (in order of position), positive integers with $y_1 + \cdots + y_k \leq m$. The map $(y_1, \ldots, y_k) \mapsto \{y_1,\ y_1+y_2,\ \ldots,\ y_1+\cdots+y_k\}$ sends such a tuple to a k-element subset of $[m]$, since the partial sums are strictly increasing (each $y_i \geq 1$) and the last one is at most m; its inverse sends $\{s_1 < \cdots < s_k\}$ to the tuple of consecutive differences $(s_1, s_2 - s_1, \ldots, s_k - s_{k-1})$, which are positive integers with sum $s_k \leq m$. So there are $\binom{m}{k}$ choices of absolute values. Since a tuple determines its positions, signs and absolute values, and these three choices are independent,

$$f(m,n) = \sum_{k \geq 0} \binom{n}{k} 2^k \binom{m}{k},$$

where terms with $k > \min(m,n)$ vanish. This expression is symmetric in m and n, so $f(m,n) = f(n,m)$. □

Remark. Two other routes. *Recursion:* split the tuples counted by $f(m,n)$ into those with $\sum |x_i| \leq m-1$, those with $\sum |x_i| = m$ and $x_n \geq 0$, and those with $\sum |x_i| = m$ and $x_n < 0$; dropping x_n (which is determined by the other coordinates in the last two cases) gives bijections with the sets counted by $f(m-1,n)$, $f(m,n-1)$, $f(m-1,n-1)$, so $f(m,n) = f(m-1,n) + f(m,n-1) + f(m-1,n-1)$ with $f(m,0) = f(0,n) = 1$, and symmetry follows by induction on $m+n$. *Generating function:* $\sum_{x \in \mathbb{Z}} t^{|x|} = \frac{1+t}{1-t}$, so $\sum_{m,n} f(m,n) t^m y^n = \frac{1}{1-t} \cdot \frac{1}{1-y\frac{1+t}{1-t}} = \frac{1}{1-t-y-ty}$, visibly symmetric in t and y.

Problem 6. $\star\star\star\star$ *(Putnam 2015 B5) P_n counts permutations π of $[n]$ with $|\pi(i) - \pi(i+1)| \leq 2$ for all i. Show $P_{n+5} - P_{n+4} - P_{n+3} + P_n$ is constant for $n \geq 2$ and find it.*

Key idea. The value n can only sit next to $n-1$ and $n-2$; classify sequences by where n sits, and relate length $n+1$ to length n by deleting $n+1$.

Solution. The constant is 4. Identify π with the sequence $(\pi(1), \ldots, \pi(n))$, a listing of $[n]$ in which consecutive terms differ by at most 2; call such a listing *admissible*. Reversal preserves admissibility. In an admissible listing of $[n]$ with $n \geq 3$, the neighbors of the term n lie in $\{n-1, n-2\}$: so either n is at one end, or n is interior and flanked by $n-1$ and $n-2$ (in some order).

Five classes. For $n \geq 3$ let U_n, V_n, X_n be the numbers of admissible listings ending with $(n-1,n)$, with $(n,n-1)$, with $(n-2,n)$, respectively; split $X_n = W_n + T_n$ according to whether the listing begins with $n-1$ (W_n) or not (T_n). Let S_n be the number of admissible listings in which $n-1$ is immediately followed by n, with $n-1$ not in the first position and n not in the last. We claim

$$P_n = 2\,(U_n + V_n + W_n + T_n + S_n) \qquad (n \geq 3).$$

Indeed, partition the admissible listings by the location of n. Those with n last end with $(n-1,n)$ or $(n-2,n)$: there are $U_n + W_n + T_n$ of them, and reversal matches them with those having n first.

Those with n interior contain the block $(n-1, n, n-2)$ or its reverse $(n-2, n, n-1)$. A listing containing $(n-2, n, n-1)$ with $n-1$ last is exactly one counted by V_n; reversal matches these with the listings containing $(n-1, n, n-2)$ with $n-1$ first. The remaining listings containing $(n-1, n, n-2)$, those with $n-1$ not first, are exactly those counted by S_n (the block forces n not to be last), and reversal matches them with the remaining listings containing $(n-2, n, n-1)$. Adding up gives the claim.

Recurrences. Deleting the term $n+1$ from an admissible listing of $[n+1]$ gives an admissible listing of $[n]$: if $n+1$ was at an end nothing else changes, and if it was interior its neighbors n and $n-1$ become adjacent, which is allowed. For $n \geq 3$:

(i) $U_{n+1} = U_n + W_n + T_n$: listings ending $(n, n+1)$ correspond, by deleting $n+1$, to listings of $[n]$ ending in n, which end with $(n-1, n)$ or $(n-2, n)$; conversely appending $n+1$ to any listing ending in n is admissible.

(ii) $V_{n+1} = U_n$: a listing ending $(n+1, n)$ has $n+1$ interior, so it ends $(n-1, n+1, n)$; deleting $n+1$ gives a listing ending $(n-1, n)$, and inserting $n+1$ between the last two terms of such a listing inverts this.

(iii) $W_n = 1$: reversing, W_n counts admissible listings that start with n and end with $n-1$ (the second term is then automatically $n-2$, since $n-1$ is last and $n \geq 3$). For $n = 3$ the only one is $(3, 1, 2)$. For $n \geq 4$, deleting the first term n leaves an admissible listing of $[n-1]$ that starts with $n-2$ and ends with $n-1$, the reverse of a listing of $[n-1]$ starting with $n-1$ and ending with $n-2$; by induction there is exactly one, and prepending n to its reverse is admissible. So $W_n = 1$ for all $n \geq 3$.

(iv) $T_{n+1} = V_n$: a listing ending $(n-1, n+1)$ and not starting with n becomes, after deleting $n+1$, a listing of $[n]$ ending in $n-1$ and not starting with n. In it, n is neither first nor last, so it is flanked by $n-1$ and $n-2$; as $n-1$ is last, the listing ends $(n-2, n, n-1)$ and is counted by V_n. Conversely a listing counted by V_n does not start with n (its n is in position $n-1 \geq 2$), and appending $n+1$ after the final $n-1$ is admissible.

(v) $S_{n+1} = S_n + V_n$: a listing counted by S_{n+1} contains $(n, n+1, n-1)$ with n not first. Deleting $n+1$ gives a listing of $[n]$ in which n is immediately followed by $n-1$ and n is not first; then n is interior, so the listing contains $(n-2, n, n-1)$. Either $n-1$ is last (counted by V_n) or not (the reverse then contains $(n-1, n, n-2)$ with $n-1$ not first, and is counted by S_n). Conversely, inserting $n+1$ between n and $n-1$ in any such listing gives one counted by S_{n+1}.

For $n = 3$ the six listings give $(U_3, V_3, W_3, T_3, S_3) = (1, 1, 1, 0, 0)$, and the recurrences produce the table

n	U_n	V_n	W_n	T_n	S_n	P_n
3	1	1	1	0	0	6
4	2	1	1	1	1	12
5	4	2	1	1	2	20
6	6	4	1	2	4	34
7	9	6	1	4	8	56
8	14	9	1	6	14	88

with $P_2 = 2$ directly. (The reader can confirm $P_4 = 12$ by listing the four admissible listings starting with 1, the two starting with 2, the two starting with 3, and the four starting with 4.)

Solving. Put $Q_n = P_n/2$. Adding (i)–(v) for $n \geq 3$,

$$Q_{n+1} = (U_n + 1 + T_n) + U_n + 1 + V_n + (S_n + V_n) = Q_n + U_n + V_n + 1 \qquad (n \geq 3),$$

and hence, using $V_n = U_{n-1}$ from (ii), $Q_{n+1} = Q_n + U_n + U_{n-1} + 1$ for $n \geq 4$. Also, from (i), (iii), (iv), (ii): $U_{n+1} = U_n + 1 + T_n = U_n + 1 + V_{n-1} = U_n + U_{n-2} + 1$ for $n \geq 5$. Set $D_n = Q_{n+1} - Q_n - 1 = U_n + U_{n-1}$ for $n \geq 4$. Then for $n \geq 6$,

$$D_{n+1} - D_n - D_{n-2} = (U_{n+1} - U_n - U_{n-2}) + (U_n - U_{n-1} - U_{n-3}) = 1 + 1 = 2.$$

Writing this out in terms of Q, the three -1's contribute $-1 + 1 + 1 = 1$, so

$$Q_{n+2} - 2Q_{n+1} + Q_n - Q_{n-1} + Q_{n-2} = 1 \qquad (n \geq 6).$$

Now add this relation for n and for $n+1$: the terms $Q_n - Q_n$ and $-Q_{n-1} + Q_{n-1}$ cancel, $Q_{n+2} - 2Q_{n+2}$ and $-2Q_{n+1} + Q_{n+1}$ combine, and we are left with

$$Q_{n+3} - Q_{n+2} - Q_{n+1} + Q_{n-2} = 2 \qquad (n \geq 6),$$

i.e. $P_{m+5} - P_{m+4} - P_{m+3} + P_m = 4$ for all $m \geq 4$. The two remaining cases come from the table: $P_7 - P_6 - P_5 + P_2 = 56 - 34 - 20 + 2 = 4$ and $P_8 - P_7 - P_6 + P_3 = 88 - 56 - 34 + 6 = 4$. Hence the quantity equals 4 for every $n \geq 2$. □

Remark. In operator language, with E the shift $n \mapsto n + 1$: the relations say $(E^3 - E^2 - 1)U = 1$ and $(E - 1)Q = (E + 1)U + 1$ up to index shifts, and $x^5 - x^4 - x^3 + 1 = (x - 1)(x + 1)(x^3 - x^2 - 1)$; this is why the problem's combination is exactly the one that reduces to a constant. The sequence begins $1, 2, 6, 12, 20, 34, 56, 88, 136, 208$.