Week 5: Combinatorics: Bijections, Double Counting, Recursions, Generating Functions

Lecture notes · Thu Oct 29 / Fri Oct 30

Toolbox: four ways to count.

1. **Bijection.** $|A| = |B|$ once you write down a map $A \to B$ *and* its inverse. To count A, find a B you already know: k-subsets of $[n]$, $\binom{n}{k}$; E/N lattice paths $(0,0) \to (a,b)$, $\binom{a+b}{a}$; compositions of n into k positive parts $\leftrightarrow$ $(k-1)$-subsets of $[n-1]$, $\binom{n-1}{k-1}$; multisets of size k from $[n] \leftrightarrow x_1 + \cdots + x_n = k$ in $\mathbb{Z}_{\geq 0} \leftrightarrow k$ stars and $n-1$ bars, $\binom{n+k-1}{k}$.
2. **Double counting.** If $I \subseteq A \times B$, then $\sum_{a\in A} \#\{b : (a,b) \in I\} = |I| = \sum_{b\in B} \#\{a : (a,b) \in I\}$: row sums equal column sums of the incidence matrix. Handshake lemma $\sum_v \deg v = 2|E|$; "count the pairs (element, set containing it)." The payoff is often an inequality, after bounding one side.
3. **Recursion.** Condition on one feature (is n in the set? what is the last step? where is the largest element?) to express a_n through smaller cases; e.g. derangements satisfy $D_n = (n-1)(D_{n-1} + D_{n-2})$. Record the base cases *and* the range of n where the recurrence is valid. The recurrence $a_n = c_1 a_{n-1} + \cdots + c_k a_{n-k}$ has solutions r^n, r a root of $x^k - c_1 x^{k-1} - \cdots - c_k$.
4. **Generating functions.** $A(x) = \sum a_n x^n$ is a formal power series; convergence is irrelevant. Products encode convolutions: independent choices multiply, so $c_n = \sum_k a_k b_{n-k}$ means $C = AB$. Know $\frac{1}{1-x} = \sum x^n$, $\frac{1}{(1-x)^k} = \sum_n \binom{n+k-1}{k-1} x^n$, $(1+x)^n = \sum \binom{n}{k} x^k$, $\frac{1}{1-x-x^2} = \sum F_{n+1} x^n$. Rational generating function = linear recurrence with constant coefficients.

Example 1 (Double counting: committees with a chair)**.** $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$. *Count pairs (S,s) with $S \subseteq [n]$ and $s \in S$. Grouping by $|S| = k$ gives $\sum_k \binom{n}{k} \cdot k$. Choosing the chair s first (n ways) and then any subset of the other $n-1$ elements gives $n2^{n-1}$. The same move proves $\sum_k \binom{n}{k}^2 = \binom{2n}{n}$: choose n people from n red and n blue, grouped by the number k of reds; in generating-function language, $[x^n]\,(1+x)^n(1+x)^n$.*

Example 2 (Bijection: the reflection principle)**.** The number of E/N lattice paths from $(0,0)$ to (n,n) that never rise above the diagonal $y = x$ is $\binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n}$ (the Catalan number). *Call a path* bad *if it touches the line $y = x+1$. Reflect the initial segment of a bad path, up to its first point on that line, across the line: $(a,b) \mapsto (b-1, a+1)$, so $(0,0) \mapsto (-1,1)$ and the result is a path from $(-1,1)$ to (n,n). Conversely, every path from $(-1,1)$ to (n,n) starts above $y = x+1$ and ends below it, and $y - x$ changes by ± 1 per step, so it touches the line; reflecting its initial segment back recovers a bad path. So the bad paths are in bijection with the paths from $(-1,1)$ to (n,n), which use $n+1$ east and $n-1$ north steps: $\binom{2n}{n+1}$ of them. The lesson: to count a complement, map the bad objects bijectively onto something unconstrained.*

Example 3 (Generating functions: Euler's partition identity)**.** The number of partitions of n into distinct parts equals the number of partitions of n into odd parts. *Each part size k is used or not, so the distinct-part partitions have generating function $\prod_{k\geq 1}(1+x^k)$. Each odd k may be used any number of times, giving $\prod_{k \text{ odd}} \frac{1}{1-x^k}$ for the odd-part partitions. Since $1 + x^k = \frac{1-x^{2k}}{1-x^k}$,*

$$\prod_{k\geq 1}(1+x^k) = \prod_{k\geq 1}\frac{1-x^{2k}}{1-x^k} = \frac{(1-x^2)(1-x^4)(1-x^6)\cdots}{(1-x)(1-x^2)(1-x^3)\cdots} = \prod_{k \text{ odd}}\frac{1}{1-x^k},$$

because every even factor $1 - x^{2k}$ in the denominator cancels against the numerator. Compare coefficients of x^n. No bijection was needed, though one exists (write each multiplicity in binary).

Pitfall. (1) "Every X produces a Y" is a map, not a bijection; write the inverse and check it lands in the right set (the reflection above works only because the inverse lands in the *bad* paths). (2) Ordered versus unordered: divide by $k!$ only when all $k!$ rearrangements are genuinely distinct. (3) A recurrence proved for $n \geq 3$ says nothing at $n = 2$; degenerate small cases (a 3×1 board, a permutation of $\{1, 2\}$) are where the edge cases hide. (4) Formal series may be multiplied freely and divided when the constant term is nonzero, but never substitute $x = 1$ unless the series is a polynomial or you have checked convergence there.

How to recognize this technique on the exam. "Find the number of..." and "show that the number of X equals the number of Y" are the flags; the shape of the answer picks the tool. A closed form like $\binom{a}{b}$ or 2^{n-2} wants a bijection with subsets or paths; a Fibonacci-like sequence or a relation among $a_{n+5}, a_{n+4}, a_{n+3}, a_n$ wants a recursion from conditioning on the largest element or the last step; an inequality between counts (edges, incidences) wants double counting plus one estimate; a count that is a polynomial in n, symmetric in two parameters, or a sum of products wants a generating function. Compute the first four or five values by hand first: they suggest the pattern and check the answer.

Suggested timing (15 minutes). 0–4: toolbox items 1–2 and Example 1 (count pairs two ways). 4–8: Example 2, the reflection principle; stress "inverse lands in the right set." 8–11: recursions, conditioning on the largest element, base cases and validity range. 11–14: Example 3, generating functions as bookkeeping for independent choices. 14–15: the recognition paragraph; then groups start on Problems 1–4.