Week 5: Combinatorics: Bijections, Double Counting, Recursions, Generating Functions

Handout · Thu Oct 29 / Fri Oct 30

> **Bijection:** to count A, map it invertibly onto something you can count (subsets, lattice paths, compositions, stars and bars); write the inverse. **Double counting:** count pairs (a, b) with $a \sim b$ two ways, $\sum_a \deg a = \sum_b \deg b$; the handshake lemma $\sum_v \deg v = 2|E|$ is the model. **Recursion:** condition on the largest element or the last step to get a_n from smaller cases; record base cases and the range where the recurrence holds. **Generating functions:** $\sum a_n x^n$, formal; independent choices multiply, $\frac{1}{(1-x)^k} = \sum_n \binom{n+k-1}{k-1} x^n$, rational series $\Leftrightarrow$ linear recurrence. Compute the first few values by hand before choosing a tool.

Problem 1. $\star$ *(warm-up)* Let a_n be the number of subsets of $\{1, 2, \ldots, n\}$ (the empty set included) that contain no two consecutive integers.

(a) Find a recurrence for a_n and deduce $a_n = F_{n+2}$, where $F_1 = F_2 = 1$ and $F_{k+1} = F_k + F_{k-1}$.

(b) Show that exactly $\binom{n-k+1}{k}$ of these subsets have k elements, and conclude that $F_{n+2} = \sum_{k \geq 0} \binom{n-k+1}{k}$.

Problem 2. $\star$ *(Mantel, 1907)* Let G be a graph with n vertices (no loops or multiple edges) that contains no triangle, i.e. no three mutually adjacent vertices. Prove that G has at most $n^2/4$ edges. *Hint: if uv is an edge, what can you say about $\deg u + \deg v$? Add this up over all edges.*

Problem 3. $\star\star$ *(Putnam 2003 A1)* Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers,

$$n = a_1 + a_2 + \cdots + a_k,$$

with k an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$? For example, with $n = 4$ there are four ways: 4, $2+2$, $1+1+2$, $1+1+1+1$.

Problem 4. $\star\star$ *(Putnam 2005 A2)* Let $S = \{(a, b) : a = 1, 2, \ldots, n,\ b = 1, 2, 3\}$. A *rook tour* of S is a polygonal path made up of line segments connecting points $p_1, p_2, \ldots, p_{3n}$ in sequence such that (i) $p_i \in S$, (ii) p_i and p_{i+1} are a unit distance apart, for $1 \leq i < 3n$, (iii) for each $p \in S$ there is a unique i such that $p_i = p$. How many rook tours are there that begin at $(1, 1)$ and end at $(n, 1)$?

Problem 5. $\star\star\star$ *(Putnam 2005 B4)* For positive integers m and n, let $f(m, n)$ denote the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \leq m$. Show that $f(m, n) = f(n, m)$.

Problem 6. $\star\star\star\star$ *(Putnam 2015 B5)* Let P_n be the number of permutations π of $\{1, 2, \ldots, n\}$ such that

$$|i - j| = 1 \quad \text{implies} \quad |\pi(i) - \pi(j)| \leq 2$$

for all i, j in $\{1, 2, \ldots, n\}$. Show that for $n \geq 2$, the quantity

$$P_{n+5} - P_{n+4} - P_{n+3} + P_n$$

does not depend on n, and find its value.

Problems 1–4 are the in-session targets; 5–6 are take-home. Solutions will be posted after Friday's session.