Week 3: Polynomials

Solutions · Thu Oct 8 / Fri Oct 9

Problem 1. ⋆ *(warm-up) For $p \in \mathbb{Z}[x]$: (a) $(a-b) \mid p(a)-p(b)$ for integers $a \neq b$; (b) there are no distinct integers a, b, c with $p(a)=b$, $p(b)=c$, $p(c)=a$.*
Key idea. $a-b$ divides a^m-b^m for every m, and divisibility is inherited by integer linear combinations.
Solution. (a) For each $m \geq 1$ we have the identity $a^m - b^m = (a-b)(a^{m-1} + a^{m-2}b + \cdots + b^{m-1})$, and the second factor is an integer, so $(a-b) \mid a^m - b^m$; for $m = 0$ the difference is 0. Writing $p(x) = \sum_m c_m x^m$ with $c_m \in \mathbb{Z}$,

$$p(a) - p(b) = \sum_m c_m \, (a^m - b^m)$$

is an integer combination of multiples of $a-b$, hence a multiple of $a-b$.

(b) Suppose such a, b, c exist. Applying (a) three times,

$$(a-b) \mid p(a)-p(b) = b-c, \qquad (b-c) \mid p(b)-p(c) = c-a, \qquad (c-a) \mid p(c)-p(a) = a-b.$$

All three differences are nonzero, and a nonzero integer that divides another has absolute value at most as large, so $|a-b| \leq |b-c| \leq |c-a| \leq |a-b|$. Thus $|a-b| = |b-c| = |c-a| = d$ for some integer $d \geq 1$. Now the three differences are each $\pm d$ and they sum to $(a-b)+(b-c)+(c-a) = 0$. But a sum of three terms each equal to $\pm d$ is $d \cdot (\pm 1 \pm 1 \pm 1)$, an *odd* multiple of d, which is nonzero. Contradiction. □

Problem 2. ⋆ *(classical) p has degree n and $p(k) = k/(k+1)$ for $k = 0, 1, \ldots, n$. Find $p(n+1)$.*
Key idea. Clear the denominator to get a polynomial that vanishes at the $n+1$ known points, then pin down its constant by evaluating at one more convenient point.
Solution. Set $q(x) = (x+1)p(x) - x$. This is a polynomial of degree at most $n+1$, and for $k = 0, 1, \ldots, n$ we have $q(k) = (k+1) \cdot \frac{k}{k+1} - k = 0$. By the factor theorem applied $n+1$ times, q is divisible by $x(x-1)\cdots(x-n)$, and comparing degrees the quotient is a constant:

$$q(x) = c\, x(x-1)(x-2)\cdots(x-n) \qquad \text{for some constant } c$$

(possibly $c = 0$ at this point). Now evaluate at $x = -1$, where the p-term disappears: $q(-1) = 0 \cdot p(-1) - (-1) = 1$, while the right side is $c\,(-1)(-2)\cdots(-(n+1)) = c\,(-1)^{n+1}(n+1)!$. Hence $c = (-1)^{n+1}/(n+1)!$.

Finally evaluate at $x = n+1$: $q(n+1) = (n+2)p(n+1) - (n+1)$, while the right side is $c\,(n+1)n\cdots 1 = c\,(n+1)! = (-1)^{n+1}$. Therefore

$$p(n+1) = \frac{(n+1) + (-1)^{n+1}}{n+2} = \begin{cases} 1, & n \text{ odd}, \\ \dfrac{n}{n+2}, & n \text{ even}. \end{cases}$$

(Sanity check with $n = 1$: $p(0) = 0$, $p(1) = \frac{1}{2}$ gives $p(x) = x/2$, and $p(2) = 1$.) □

Problem 3. ⋆⋆ *(Putnam 2023 A2) n is even, p is monic of degree $2n$, and $p(1/k) = k^2$ for all integers $1 \leq |k| \leq n$. Find all other real x with $p(1/x) = x^2$.*
Key idea. The condition $p(1/x) = x^2$ says that $y = 1/x$ is a root of $y^2 p(y) - 1$, a polynomial of degree $2n+2$ of which we already know $2n$ roots.

Solution. For $x \neq 0$ (the equation is meaningless at $x = 0$), put $y = 1/x$. Then $p(1/x) = x^2$ is equivalent to $p(y) = 1/y^2$, i.e. to $q(y) = 0$ where

$$q(y) = y^2 p(y) - 1.$$

Since p is monic of degree $2n$, q is monic of degree $2n + 2$, and the hypothesis says $q(1/k) = 0$ for the $2n$ distinct numbers $1/k$, $1 \leq |k| \leq n$. By the factor theorem and a degree count, q is the product of these $2n$ linear factors and a monic quadratic:

$$q(y) = (y^2 + cy + d)\, E(y), \qquad E(y) = \prod_{k=1}^{n} \Big(y - \frac{1}{k}\Big)\Big(y + \frac{1}{k}\Big) = \prod_{k=1}^{n} \Big(y^2 - \frac{1}{k^2}\Big).$$

The polynomial E is even (only even powers of y appear), and $E(0) = \prod_{k=1}^{n}(-1/k^2) = (-1)^n/(n!)^2 = 1/(n!)^2$ because n is even.

Now compare two coefficients of $q(y) = y^2 p(y) - 1$, which has no y^1 term and constant term -1. The coefficient of y^1 in $(y^2 + cy + d)E(y)$ is $c\,E(0) + d \cdot [\text{coefficient of } y^1 \text{ in } E] = c\,E(0)$, so $c\,E(0) = 0$ and $c = 0$. The constant term is $d\,E(0) = d/(n!)^2$, so $d = -(n!)^2$. Hence

$$q(y) = \big(y^2 - (n!)^2\big) \prod_{k=1}^{n} \Big(y^2 - \frac{1}{k^2}\Big),$$

whose roots are exactly $y = \pm 1/k$ $(1 \leq k \leq n)$ and $y = \pm n!$. Translating back through $x = 1/y$: the real solutions of $p(1/x) = x^2$ are $x = \pm 1, \ldots, \pm n$ (given) and two new ones (new because $n \geq 2$, so $0 < 1/n! \leq \frac{1}{2}$ is not among $\pm 1, \ldots, \pm n$):

$$x = \frac{1}{n!} \quad \text{and} \quad x = -\frac{1}{n!}.$$

This holds for every p satisfying the hypotheses: the quadratic factor was forced by the coefficient comparison, so the answer does not depend on which such p is chosen. □

Remark. If n were odd, $E(0) = -1/(n!)^2$ and the same computation gives $d = +(n!)^2$; the quadratic $y^2 + (n!)^2$ then has no real roots, so there would be no further real solutions at all. That is why the problem insists on n even.

Problem 4. ⋆⋆ *(Putnam 2024 A2) Find all real polynomials p for which some real polynomial q satisfies $p(p(x)) - x = (p(x) - x)^2 q(x)$.*

Key idea. Write $p(x) = x + r(x)$; expanding $p(x + r)$ to first order in r turns the condition into "r divides $1 + p'$," which a degree comparison kills.

Solution. Answer: exactly the polynomials $p(x) = x + c$ and $p(x) = -x + c$, $c \in \mathbb{R}$.

These work. If $p(x) = x + c$ then $p(p(x)) - x = 2c$ and $(p(x) - x)^2 = c^2$; take $q = 2/c$ if $c \neq 0$ and $q = 0$ if $c = 0$. If $p(x) = -x + c$ then $p(p(x)) = -(-x + c) + c = x$, so the left side is 0 and $q = 0$ works.

Nothing else works. Let $r(x) = p(x) - x$, so the condition is $r(x)^2 \mid p(p(x)) - x$ in $\mathbb{R}[x]$. We use the first-order expansion: for any polynomial p and any polynomial h,

$$p(x + h) \equiv p(x) + h\,p'(x) \pmod{h^2},$$

because $(x + h)^m = x^m + m x^{m-1} h + h^2(\cdots)$ by the binomial theorem, and the claim is linear in p. Applying this with $h = r(x)$, and noting $x + r(x) = p(x)$,

$$p(p(x)) - x \equiv p(x) + r(x)p'(x) - x = r(x)\big(1 + p'(x)\big) \pmod{r(x)^2}.$$

So the hypothesis $r^2 \mid p(p(x)) - x$ is equivalent to $r^2 \mid r\,(1 + p')$. If $r = 0$ then $p(x) = x$, which is on our list. Otherwise cancel one factor of r (legitimate in the integral domain $\mathbb{R}[x]$): $r \mid 1 + p'$.

Now compare degrees. If $\deg p \geq 2$, then $\deg r = \deg p$ while $\deg(1 + p') = \deg p - 1$; a nonzero polynomial cannot be divisible by one of larger degree, so $1 + p' = 0$, which forces $\deg p \leq 1$, a contradiction. Hence $\deg p \leq 1$: write $p(x) = ax + b$. Then $r(x) = (a - 1)x + b$ and $1 + p' = 1 + a$. If $a \neq 1$ then r has degree 1 and divides the constant $1 + a$, which is only possible if $1 + a = 0$, i.e. $a = -1$. So $a = 1$ or $a = -1$, giving $p(x) = x + b$ or $p(x) = -x + b$ as claimed. (A constant $p = b$ has $a = 0$ and is excluded by the same step.) □

Remark. Another route: differentiate the identity and reduce modulo r. Since $p(x) \equiv x \pmod r$, one gets $p'(x)p'(p(x)) - 1 \equiv p'(x)^2 - 1 = (p' - 1)(p' + 1) \pmod r$, and the right side of the original identity has derivative divisible by r. With $p' - 1 = r'$ this says $r \mid r'(r' + 2)$; an argument on multiplicities of roots shows every root of r is simple and is a root of $r' + 2$, and again $\deg(r' + 2) < \deg r$ forces $r' + 2 = 0$ or r constant.

Problem 5. ⋆⋆⋆ *(Putnam 2018 B2) $f_n(z) = n + (n - 1)z + \cdots + z^{n-1}$ has no roots with $|z| \leq 1$.*
Key idea. Multiply by $1 - z$ to telescope the coefficients into $n - (z + z^2 + \cdots + z^n)$, then use the equality case of the triangle inequality.

Solution. First, $f_n(1) = n + (n - 1) + \cdots + 1 = n(n + 1)/2 \neq 0$, so $z = 1$ is not a root. For every z,

$$(1 - z)f_n(z) = \sum_{j=0}^{n-1}(n - j)z^j - \sum_{j=1}^{n}(n - j + 1)z^j = n - \sum_{j=1}^{n} z^j,$$

since for $1 \leq j \leq n - 1$ the coefficient of z^j is $(n - j) - (n - j + 1) = -1$, and the z^n coefficient is -1.

Now suppose $|z| \leq 1$ and $f_n(z) = 0$. Then $z \neq 1$ and $(1 - z)f_n(z) = 0$, so

$$z + z^2 + \cdots + z^n = n.$$

By the triangle inequality, $|z + z^2 + \cdots + z^n| \leq |z| + |z|^2 + \cdots + |z|^n \leq n$, with equality in the second step only if $|z| = 1$, and equality in the first step only if the summands $z, z^2, \ldots, z^n$ all point in the same direction (each is a nonnegative real multiple of a common unit vector). Since their sum is the positive real number n, that common direction is the positive real axis, so in particular z itself is a nonnegative real number. Together with $|z| = 1$ this gives $z = 1$, contradicting $z \neq 1$. Hence f_n has no root in the closed unit disk. □

Remark. Reciprocal-polynomial route: $g_n(z) = z^{n-1}f_n(1/z) = 1 + 2z + \cdots + nz^{n-1}$ is the derivative of $1 + z + \cdots + z^n$, whose roots are the $(n + 1)$-st roots of unity other than 1. The Gauss–Lucas theorem places the roots of a derivative in the convex hull of the roots, and (as the roots are simple) not at the vertices, so all roots of g_n satisfy $|z| < 1$; hence all roots of f_n satisfy $|z| > 1$.

Problem 6. ⋆⋆⋆⋆ *(Putnam 2014 B4) All roots of $P_n(x) = \sum_{k=0}^{n} 2^{k(n-k)}x^k$ are real.*
Key idea. At $x = -2^{2j-n}$ the exponent of the k-th term becomes $j^2 - (k - j)^2$, so the $k = j$ term carries the sign $(-1)^j$ and the rest is a smaller correction; $n + 1$ alternating signs give n real roots by the intermediate value theorem.

Solution. For $n = 1$, $P_1(x) = 1 + x$, and for $n = 2$, $P_2(x) = 1 + 2x + x^2 = (1 + x)^2$; both have only real roots. Assume $n \geq 3$. Note that P_n has degree exactly n (leading coefficient $2^0 = 1$), so it suffices to exhibit n distinct real roots.

For $j = 0, 1, \ldots, n$ set $x_j = -2^{2j-n}$. The k-th term of $P_n(x_j)$ is

$$2^{k(n-k)}\,(-1)^k\,2^{k(2j-n)} = (-1)^k\,2^{2jk-k^2} = (-1)^k\,2^{j^2-(k-j)^2}.$$

Substituting $d = k - j$ (so d runs over $-j, \dots, n-j$ and $(-1)^k = (-1)^j(-1)^d$),

$$P_n(x_j) = (-1)^j\, 2^{j^2}\, S_j, \qquad S_j = \sum_{d=-j}^{n-j} (-1)^d\, 2^{-d^2}.$$

Claim: $S_j > 0$. Split $S_j = T_0 + T_+ + T_-$, where T_0 collects the terms with $|d| \le 1$, T_+ those with $d \ge 2$, and T_- those with $d \le -2$.

The terms of T_+, in order $d = 2, 3, 4, 5, \dots$, are $+2^{-4}, -2^{-9}, +2^{-16}, -2^{-25}, \dots$. Pair each even $d = 2m$ with the following odd $d = 2m+1$: each pair contributes $2^{-4m^2} - 2^{-(2m+1)^2} > 0$. An odd d is never left unpaired (its even predecessor $d - 1 \ge 2$ is in the range whenever d is), while an unpaired even d contributes a positive term. So $T_+ \ge 0$, and $T_+ > 0$ whenever the range contains some $d \ge 2$, i.e. whenever $n - j \ge 2$. By the same pairing on the negative side, $T_- \ge 0$, with $T_- > 0$ whenever $j \ge 2$.

For T_0: the term $d = 0$ contributes 1; the terms $d = 1$ and $d = -1$ contribute $-\frac{1}{2}$ each, and are present iff $n - j \ge 1$, resp. $j \ge 1$. Hence $T_0 \ge 0$, and $T_0 = 0$ only when $j \ge 1$ and $n - j \ge 1$. In that case, since $j + (n - j) = n \ge 3$, at least one of $j \ge 2$ or $n - j \ge 2$ holds, so $T_+ + T_- > 0$. In every case $S_j > 0$, proving the claim.

Therefore $P_n(x_j)$ has the sign of $(-1)^j$ for $j = 0, 1, \dots, n$. The points $x_0 = -2^{-n} > x_1 > \cdots > x_n = -2^n$ are distinct, and P_n changes sign on each of the n pairwise disjoint open intervals (x_{j+1}, x_j), $0 \le j \le n-1$. By the intermediate value theorem each contains a root, so P_n has n distinct real roots. Since $\deg P_n = n$, these are all of its roots. □

Remark. Why these evaluation points? On a logarithmic scale the coefficients $2^{k(n-k)}$ form a parabola in k; choosing $x = -2^{2j-n}$ shifts the parabola so that its peak sits exactly at $k = j$, and the alternating signs then make the peak and its two neighbours cancel exactly, while everything further out contributes a small positive correction. The same idea, "choose x so that a single term of the sum dominates," proves real-rootedness for many sums with rapidly growing coefficients.