Week 3: Polynomials

Handout · Thu Oct 8 / Fri Oct 9

Toolbox. A nonzero polynomial of degree n has at most n roots, so two polynomials of degree $\leq n$ that agree at $n+1$ points are equal, and a polynomial with infinitely many roots is zero; a degree-$\leq n$ polynomial is pinned down by $n+1$ values (interpolation), and the useful move is to build an auxiliary polynomial that *vanishes* at the known points. If $p \in \mathbb{Z}[x]$ then $(a-b) \mid p(a)-p(b)$ for integers $a \neq b$, and $p(x) \mp 1$ factors in $\mathbb{Z}[x]$ through its known roots. Vieta: the elementary symmetric functions of the roots are $\pm$ the coefficients. Degree/growth: compare degrees and leading terms first; $r \mid s \neq 0$ forces $\deg r \leq \deg s$; $p(x+h) \equiv p(x) + h\,p'(x) \pmod{h^2}$. Roots are located by sign changes (real) or by the triangle inequality and the reciprocal polynomial $x^n p(1/x)$ (complex).

Problem 1. $\star$ *(warm-up)* Let p be a polynomial with integer coefficients.

(a) Show that $(a-b)$ divides $p(a)-p(b)$ for all integers $a \neq b$.

(b) Show that there are no distinct integers a, b, c with $p(a) = b$, $p(b) = c$, and $p(c) = a$.

Problem 2. $\star$ *(classical)* Let $n \geq 1$ and let p be a polynomial of degree n such that

$$p(k) = \frac{k}{k+1} \qquad \text{for } k = 0, 1, 2, \ldots, n.$$

Find $p(n+1)$.

Problem 3. $\star\star$ *(Putnam 2023 A2)* Let n be an even positive integer, and let p be a monic real polynomial of degree $2n$; that is, $p(x) = x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1 x + a_0$ for some real numbers $a_0, \ldots, a_{2n-1}$. Suppose that $p(1/k) = k^2$ for every integer k with $1 \leq |k| \leq n$. Find all other real numbers x for which $p(1/x) = x^2$.

Problem 4. $\star\star$ *(Putnam 2024 A2)* For which real polynomials p is there a real polynomial q such that

$$p(p(x)) - x = \big(p(x) - x\big)^2 q(x)$$

for all real x?

Problem 5. $\star\star\star$ *(Putnam 2018 B2)* Let n be a positive integer, and let $f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}$. Prove that f_n has no roots in the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$.

Problem 6. $\star\star\star\star$ *(Putnam 2014 B4)* Show that for each positive integer n, all the roots of the polynomial

$$\sum_{k=0}^{n} 2^{k(n-k)} x^k$$

are real numbers.

Problems 1–4 are the in-session targets; 5–6 are take-home. Solutions will be posted after Friday's session.