Week 2: Invariants, Monovariants, Parity, and Coloring

Solutions · Thu Oct 1 / Fri Oct 2

Problem 1. ⋆ *(warm-up) (a) An* 8×8 *board minus two opposite corners has no domino tiling. (b) A* 10×10 *board has no tiling by* 1×4 *rectangles.*

Key idea. Color the board so that every tile covers a fixed number of cells of each color, then compare with the color counts of the region.

Solution. (a) Color the board as a chessboard. Every domino covers one black and one white square, so a tiled region has equally many black and white squares. Opposite corners of a chessboard have the same color (the corner (i, j) has color determined by $i + j \bmod 2$, and $(1, 1)$, $(8, 8)$ both have $i + j$ even). Removing them leaves 30 squares of one color and 32 of the other, so no tiling exists.

(b) Color column j with color $j \bmod 4$. The color classes have sizes $30, 30, 20, 20$ (columns $\{1, 5, 9\}$, $\{2, 6, 10\}$, $\{3, 7\}$, $\{4, 8\}$, each column holding 10 cells). A horizontal 1×4 tile covers one cell of each color; a vertical tile covers four cells of a single color. Suppose a tiling uses h horizontal tiles and v_c vertical tiles in columns of color c. Counting cells of color c gives $30 = h + 4v_1 = h + 4v_2$ and $20 = h + 4v_3 = h + 4v_0$. The first equation forces $h \equiv 2 \pmod 4$ and the second forces $h \equiv 0 \pmod 4$, a contradiction. □

Remark. In (b) the popular alternative is the four-coloring by $(i + j) \bmod 4$, which every 1×4 tile meets once in each color; the four classes then have sizes $25, 24, 25, 26$, again unequal. Either way the point is the same: find a coloring that every placement of the tile meets in a controlled way.

Problem 2. ⋆ *(warm-up) (a) Erase* a, b *from* $\{1, \ldots, 2026\}$ *and write* $|a - b|$*; show the last number is odd. (b) Negating rows or columns of a real* $n \times n$ *table, reach a table with all row and column sums nonnegative.*

Key idea. (a) The sum mod 2 is invariant. (b) The total sum is a strictly increasing monovariant on a finite set of states.

Solution. (a) Since $|a - b| \equiv a + b \pmod 2$, each move preserves the parity of the sum S of the numbers on the board. Initially $S = 1 + 2 + \cdots + 2026 = 1013 \cdot 2027$, which is odd. After 2025 moves one number remains and it equals $S \bmod 2$ in parity, so it is odd. □

(b) Let Φ be the sum of all n^2 entries. If some row has negative sum $-s < 0$, negating that row changes Φ by $+2s > 0$; the same holds for a column. So as long as a negative row or column sum exists, we can make a move that strictly increases Φ. Now the monovariant argument: every reachable table is obtained from the original by choosing a subset of rows and a subset of columns to negate, so there are at most 2^{2n} reachable tables and Φ takes at most 2^{2n} values. A strictly increasing sequence in a finite set stops, and it can only stop at a table with no negative row or column sum. □

Remark. Equivalently: among the finitely many reachable tables pick one maximizing Φ; if it had a negative row sum, negating that row would give a reachable table with larger Φ. This is the extremal principle of Week 1 in disguise.

Problem 3. ⋆⋆ *(Putnam 2024 B1) Cell* (i, j) *of an* $n \times n$ *grid holds* $i + j - k$*. For which* n, k *can one choose* n *cells, no two in a row or column, holding exactly* $1, 2, \ldots, n$*?*

Key idea. Whatever cells you pick, their total is the same; that invariant pins down k, and then a construction finishes.

Solution. **Answer:** exactly when n is odd and $k = (n + 1)/2$.

Choosing n cells with no two in the same row or column means choosing cells $(i, \sigma(i))$, $i = 1, \dots, n$, for a permutation σ of $\{1, \dots, n\}$. The sum of the chosen numbers is then

$$\sum_{i=1}^{n} \big(i + \sigma(i) - k\big) = \frac{n(n+1)}{2} + \frac{n(n+1)}{2} - nk = n(n+1) - nk,$$

independent of σ: this is the invariant. If the chosen numbers are $1, \dots, n$ in some order, the sum is $n(n+1)/2$, so $n(n+1) - nk = n(n+1)/2$, i.e. $k = (n+1)/2$. Since k is an integer, n must be odd.

Conversely let $n = 2r-1$ be odd and $k = r$. Choose $\sigma(i) = i+r-1$ for $1 \le i \le r$ and $\sigma(i) = i-r$ for $r < i \le n$. This is a permutation: the first block of values is $\{r, \dots, 2r-1\} = \{r, \dots, n\}$ and the second is $\{1, \dots, r-1\}$. The chosen numbers are $i + \sigma(i) - r = 2i - 1$ for $i \le r$, giving the odd numbers $1, 3, \dots, n$, and $i + \sigma(i) - r = 2i - 2r = 2(i-r)$ for $i > r$, giving the even numbers $2, 4, \dots, n-1$. Together these are exactly $1, \dots, n$. □

Problem 4. ⋆⋆ *(Putnam 2023 B1) Coins fill the $(m-1) \times (n-1)$ top-left block of an $m \times n$ grid; a coin at (i, j) may slide to $(i+1, j+1)$ when $(i+1, j)$, $(i, j+1)$, $(i+1, j+1)$ are empty. Count the reachable configurations.*

Key idea. The set of empty squares is always a monotone staircase path; every staircase is reachable by running the process backwards along a monovariant.

Solution. **Answer:** $\binom{m+n-2}{m-1}$.

Draw the grid with i increasing downward and j increasing to the right. Call a set E of squares a *staircase* if it can be listed as a path from $(1, n)$ to $(m, 1)$ in which every step goes down, $(i, j) \to (i+1, j)$, or left, $(i, j) \to (i, j-1)$. Such a path has exactly $m-1$ down-steps and $n-1$ left-steps in some order, so there are $\binom{m+n-2}{m-1}$ staircases. Along the path the quantity $i - j$ increases by exactly 1 per step, so a staircase contains exactly one square with $i - j = t$ for each $t \in \{1-n, \dots, m-1\}$; in particular, squares of a staircase with consecutive values of $i - j$ are consecutive on the path.

Invariant. The set E of empty squares is always a staircase. Initially E is the last column followed by the last row, $(1, n) \to \cdots \to (m, n) \to (m, n-1) \to \cdots \to (m, 1)$, the staircase with all down-steps first. Suppose E is a staircase and a coin slides from (i, j) to $(i+1, j+1)$. Then $(i, j+1), (i+1, j+1), (i+1, j) \in E$ have $i - j$ values $i - j - 1$, $i - j$, $i - j + 1$, so they occur consecutively on the path as a down-step followed by a left-step. The new empty set is $E' = E \setminus \{(i+1, j+1)\} \cup \{(i, j)\}$, whose path goes $(i, j+1) \to (i, j) \to (i+1, j)$, a left-step followed by a down-step. So E' is again a staircase. Since the coins are exactly the complement of E, a configuration is determined by its staircase, and the number of reachable configurations is at most $\binom{m+n-2}{m-1}$.

Every staircase is reachable. Let $\Phi(E) = \sum_{(i,j) \in E} (i + j)$. A legal move replaces $(i+1, j+1)$ by (i, j) in E, so it lowers Φ by 2. Now take any staircase E whose path contains a left-step followed by a down-step, say $(i, j+1) \to (i, j) \to (i+1, j)$. The square $(i+1, j+1)$ lies in the grid (since $(i+1, j), (i, j+1) \in E$) and has the same value of $i - j$ as (i, j), so it is not in E: it holds a coin. Consider the configuration with empty set $E' = E \setminus \{(i, j)\} \cup \{(i+1, j+1)\}$; it is a staircase (the corner is now down-then-left), and in it the coin at (i, j) sees $(i+1, j), (i, j+1), (i+1, j+1)$ all empty, so sliding it is a legal move that produces exactly E. Thus E is reachable in one move from E', and $\Phi(E') = \Phi(E) + 2$. Starting from an arbitrary staircase, repeat this "undo" step as long as a left-then-down corner exists. Since Φ strictly increases and is bounded (by $(m+n-1)(m+n)$), the process stops, and it stops at a staircase with no left-step followed by a down-step, i.e. all down-steps first: the initial configuration. Reversing the sequence of undo steps yields a chain of

legal moves from the initial configuration to E. Hence all $\binom{m+n-2}{m-1}$ staircases occur, and they are distinct configurations. □

Problem 5. ⋆⋆⋆ *(Putnam 2008 A3) Repeatedly pick $j < k$ with $a_j \nmid a_k$ and replace (a_j, a_k) by $(\gcd, \operatorname{lcm})$. Show the process stops and the final sequence is independent of the choices.*
Key idea. The weighted product $\prod a_i^i$ is a strictly increasing bounded integer (termination); for each prime, the multiset of exponents is invariant and the final sequence is its sorted version (uniqueness).

Solution. *Termination.* Write $g = \gcd(a_j, a_k)$ and $L = \operatorname{lcm}(a_j, a_k)$, so $gL = a_j a_k$. First, the product $P = a_1 a_2 \cdots a_n$ is invariant. Second, consider $\Psi = \prod_{i=1}^n a_i^i$. A move at (j, k) multiplies Ψ by

$$\frac{g^j L^k}{a_j^j a_k^k} = \frac{(gL)^j L^{k-j}}{(a_j a_k)^j a_k^{k-j}} = \left(\frac{L}{a_k}\right)^{k-j}.$$

Here L is a multiple of a_k, and $L \neq a_k$ because $a_j \nmid a_k$; so $L > a_k$ and, as $k > j$, the factor exceeds 1. Thus Ψ is a strictly increasing positive integer. It is bounded: each $a_i \leq P$, so $\Psi \leq P^{1+2+\cdots+n}$, and P never changes. A strictly increasing sequence of integers in a bounded range is finite, so the process stops. It can only stop when $a_j \mid a_k$ for all $j < k$.

Uniqueness. Fix a prime p and let $e_i = v_p(a_i)$ be the exponent of p in a_i. Since $v_p(\gcd) = \min$ and $v_p(\operatorname{lcm}) = \max$, a move at (j, k) replaces (e_j, e_k) by $(\min(e_j, e_k), \max(e_j, e_k))$ and leaves the other e_i alone. Hence the multiset $\{e_1, \ldots, e_n\}$ is an invariant of the process, for every prime p. In the final sequence $a_1 \mid a_2 \mid \cdots \mid a_n$, so $e_1 \leq e_2 \leq \cdots \leq e_n$; a nondecreasing sequence with a prescribed multiset of entries is unique, namely the multiset listed in increasing order. So for each prime p the final exponents $v_p(a_i)$ are determined by the initial sequence, and since $a_i = \prod_p p^{v_p(a_i)}$ (only the finitely many primes dividing P occur), the final sequence itself is determined. □

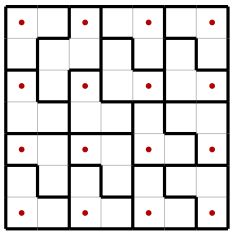
Remark. Kedlaya's termination argument is a different monovariant: a_n never decreases (it is only ever replaced by a multiple of itself) and is bounded by P, so it is eventually constant; after that no move involves position n (such a move would strictly increase a_n), so a_{n-1} never decreases, and so on down the sequence. Both arguments use the same two facts, "lcm $\geq a_k$" and "P is invariant".

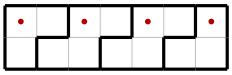
Problem 6. ⋆⋆⋆⋆ *(Putnam 2016 A4) Tile a $(2m-1) \times (2n-1)$ region, $m, n \geq 4$, with L-trominoes and S-tetrominoes (rotations and reflections allowed). Find the minimum number of tiles.*
Key idea. Color the cells with both coordinates odd: every tile covers at most one of them, and there are mn of them; then build a tiling with exactly mn tiles.

Solution. **Answer:** mn.

Lower bound (coloring). The roles of m and n are symmetric, so take the region to have $2n-1$ rows and $2m-1$ columns. Index the cells by (i, j) with $1 \leq i \leq 2n-1$ (rows) and $1 \leq j \leq 2m-1$ (columns), and color (i, j) red when i and j are both odd. There are n odd rows and m odd columns, so mn red cells. We claim no tile covers two red cells. An L-tromino lies inside some 2×2 block; a 2×2 block has exactly one odd row and one odd column, hence exactly one red cell. An S-tetromino lies inside a 2×3 block (rows $r, r+1$; columns $c, c+1, c+2$) and occupies two adjacent columns in each of the two rows, or the transposed picture inside a 3×2 block. Exactly one of the rows $r, r+1$ is odd, and in that row the tile covers two adjacent columns, exactly one of which is odd; so the tile covers exactly one red cell. (The transposed case is identical with rows and columns swapped.) Since the red cells must all be covered by distinct tiles, any tiling uses at least mn tiles.

Construction. Two building blocks, drawn with the red cells dotted so the count can be checked by eye:





Left: a 7×7 square tiled by 15 L-trominoes and one S-tetromino, $16 = 4 \cdot 4$ tiles in all ($15 \cdot 3 + 4 = 49$ cells). Right: a 2×7 strip tiled by an L-tromino, two S-tetrominoes, and an L-tromino. The strip generalizes: an L-tromino, then $k-2$ S-tetrominoes in a staircase, then an L-tromino tile a $2 \times (2k-1)$ strip with k tiles for every $k \geq 2$ (cell count $3+4(k-2)+3 = 2(2k-1)$; each S-tetromino fills the single bottom-row gap left by the previous tile and leaves a new single bottom-row gap two columns further right).

Now take the $(2n-1) \times (2m-1)$ region (rows by columns). Put the 7×7 square in the lower-left corner. To its right remain 7 rows and $2m-1-7 = 2(m-4)$ columns: cut this into $m-4$ blocks of size 7×2, each a rotated 2×7 strip using 4 tiles. Above the bottom seven rows remain $2n-1-7 = 2(n-4)$ rows of full width $2m-1$: cut this into $n-4$ strips of size $2 \times (2m-1)$, each using m tiles. The total is

$$16 + 4(m-4) + m(n-4) = mn.$$

Since $m, n \geq 4$, every piece has nonnegative size, so this is a tiling with exactly mn tiles, matching the lower bound. □

Remark. A useful sanity check on the coloring: in the 7×7 picture every tile contains exactly one dot, which is forced once we know 16 tiles cover 16 red cells. The choice of coloring is dictated by the tiles: the L-tromino "sees" one cell of a 2×2 block, so a coloring with period 2 in both directions is the natural first guess.