Week 2: Invariants, Monovariants, Parity, and Coloring

Lecture notes · Thu Oct 1 / Fri Oct 2

Toolbox.

1. **Invariant.** A quantity I(state) that every allowed move leaves unchanged. If $I(\text{start}) \neq I(\text{target})$, the target is unreachable. Standard candidates: a sum, a sum mod 2 or mod m, a product, a gcd, the sign of a permutation, the number of cells of some color.
2. **Monovariant.** A quantity Φ that *strictly* increases (or decreases) with every move and takes values in a set with no infinite increasing (decreasing) chain, e.g. the nonnegative integers or a finite set. Then the process must stop, and it stops only in a state where no move is available.
3. **Reachability = invariant + construction.** To describe *all* reachable states, find an invariant that separates the unreachable ones, then show every state with the right invariant value actually is reached (often by running the process backwards).
4. **Coloring for tilings.** Color the cells so that every placement of every tile covers a prescribed number of cells of each color (or at most one cell of a special color). Compare with the color counts of the region.
5. **Parity.** Two states differing in a parity-type invariant are never connected; a quantity that changes by ± 1 at each step differs from its initial value, after t steps, by a number with the parity of t.
6. **Extremal link.** A state that minimizes a monovariant is a state where no move applies; this is the extremal principle from Week 1 in dynamic form.

Three worked examples

Example 1 (Parity invariant)**.** *Seven cups stand upside down on a table. A move turns over exactly four cups. Can all seven end right side up? No. Let u be the number of upside-down cups. A move turns k cups from upside-down to upright and $4-k$ the other way, so u changes by $(4-k)-k = 4-2k$, an even number. Hence $u \bmod 2$ is invariant. It starts at 7 (odd) and the target has $u = 0$ (even).* □

Example 2 (Monovariant)**.** *Each vertex of a finite graph is colored red or blue. A move picks a vertex that has more neighbors of the opposite color than of its own color and recolors it. Show that the process terminates. Let Φ be the number of edges whose endpoints have different colors. If a vertex v has a neighbors of its color and $b > a$ of the other, recoloring v changes Φ by $a - b < 0$. So Φ is a strictly decreasing nonnegative integer and the process ends after at most $\Phi(start) \leq |E|$ moves.* □*Note that both ingredients matter: strictness (else the process could cycle) and integrality (a strictly decreasing sequence of positive reals need not stop).*

Example 3 (An invariant that appears after one move)**.** *Three piles have 5, 49, 51 stones. A move either merges two piles or splits a pile with an even number of stones into two equal piles. Can we reach 105 piles of one stone each? No. Claim: if an odd number d divides every pile size, this remains true forever. Merging: $d \mid a,\ d \mid b \Rightarrow d \mid a+b$. Splitting: $d \mid 2c$ and d odd $\Rightarrow d \mid c$. Initially all piles are odd, so the first move must merge two piles, producing $\{54, 51\}$, $\{56, 49\}$, or $\{100, 5\}$, whose sizes are all divisible by 3, 7, or 5 respectively. From then on that odd divisor $d > 1$ divides* *every pile, so no pile ever has size* 1. $\square$*(Lesson: the invariant need not hold at the very start; look at the state after the forced first move.)*

Pitfall. (i) A monovariant proves termination only if it is strictly monotone *and* lives in a well-founded set. "The sum decreases" is useless if the sum is a real number that can decrease by $1, \frac{1}{2}, \frac{1}{4}, \ldots$. (ii) An invariant only ever proves that something is *impossible*. If the question asks "for which states" or "how many configurations", an invariant gives an upper bound on the reachable set, and you still owe an explicit construction for the other direction. (iii) In coloring arguments, check *every* orientation and position of *every* tile; one bad placement kills the count.

How to recognize this technique on the exam

Look for a *process*: moves, operations, replacements ("replace a, b by ..."), sliding, flipping, recoloring, or a game that is really a process in disguise. Questions of the form "prove this stops", "show you cannot reach", "the final result does not depend on the choices", "how many configurations are reachable", or "what is the minimum number of tiles" are the tell-tale signs. Then ask, in this order: What is *obviously* preserved (sum, product, count mod 2)? What is monotone (a sum, a sum with weights $1, 2, \ldots, n$, the number of bad pairs)? If tiles are involved, which periodic coloring meets every tile in the same way? Write the invariant down explicitly and verify it move by move before doing anything else.

Suggested timing (15 minutes)

- 0–3 min: state the toolbox; emphasize invariant = impossibility, monovariant = termination.
- 3–10 min: Examples 1 and 2 at the board (fast), then Example 3 with students guessing the invariant.
- 10–13 min: the pitfalls, especially (i) and (ii).
- 13–15 min: preview the handout: Problems 1–2 are five-minute warm-ups; Problems 3–4 each combine an invariant with a construction (in Problem 4, by running the moves backwards).