Week 2: Invariants, Monovariants, Parity, and Coloring

Handout · Thu Oct 1 / Fri Oct 2

Toolbox. An **invariant** is a quantity no move can change; if it differs between two states, neither is reachable from the other. A **monovariant** strictly increases (or decreases) with every move and takes values in a well-founded set such as $\mathbb{Z}_{\geq 0}$; its existence forces the process to stop. For tilings, **color** the cells so every tile meets the colors in a fixed way, then count. To find *all* reachable states, pair the invariant with an explicit construction (often by running moves backwards).

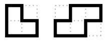
Problem 1. $\star$ *(warm-up)* (a) Two opposite corner squares are removed from an 8×8 chessboard. Show that the remaining 62 squares cannot be tiled by 1×2 dominoes. (b) Show that a 10×10 board cannot be tiled by 1×4 rectangles.

Problem 2. $\star$ *(warm-up)* (a) The numbers $1, 2, \ldots, 2026$ are written on a board. A move erases two numbers a, b and writes $|a - b|$ in their place. After 2025 moves one number remains. Show that it is odd. (b) An $n \times n$ table is filled with real numbers. A move changes the sign of every entry in one row or one column. Show that after finitely many moves one can reach a table in which every row sum and every column sum is nonnegative.

Problem 3. $\star\star$ *(Putnam 2024 B1)* Let n and k be positive integers. The square in the ith row and jth column of an $n \times n$ grid contains the number $i + j - k$. For which n and k is it possible to select n squares from the grid, no two in the same row or column, such that the numbers contained in the selected squares are exactly $1, 2, \ldots, n$?

Problem 4. $\star\star$ *(Putnam 2023 B1)* Consider an m-by-n grid of unit squares, indexed by (i, j) with $1 \leq i \leq m$ and $1 \leq j \leq n$. There are $(m-1)(n-1)$ coins, which are initially placed in the squares (i, j) with $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$. If a coin occupies the square (i, j) with $i \leq m-1$ and $j \leq n-1$ and the squares $(i+1, j)$, $(i, j+1)$, and $(i+1, j+1)$ are unoccupied, then a legal move is to slide the coin from (i, j) to $(i+1, j+1)$. How many distinct configurations of coins can be reached starting from the initial configuration by a (possibly empty) sequence of legal moves?

Problem 5. $\star\star\star$ *(Putnam 2008 A3)* Start with a finite sequence $a_1, a_2, \ldots, a_n$ of positive integers. If possible, choose two indices $j < k$ such that a_j does not divide a_k, and replace a_j and a_k by $\gcd(a_j, a_k)$ and $\operatorname{lcm}(a_j, a_k)$, respectively. Prove that if this process is repeated, it must eventually stop and the final sequence does not depend on the choices made.

Problem 6. $\star\star\star\star$ *(Putnam 2016 A4)* Consider a $(2m-1) \times (2n-1)$ rectangular region, where m and n are integers such that $m, n \geq 4$. This region is to be tiled using tiles of the two types shown here:  (the dotted lines divide the tiles into 1×1 squares). The tiles may be rotated and reflected, as long as their sides are parallel to the sides of the rectangular region. They must all fit within the region, and they must cover it completely without overlapping. What is the minimum number of tiles required to tile the region?

Problems 1–4 are the in-session targets; 5–6 are take-home. Solutions will be posted after Friday's session.