Week 1: Induction, Pigeonhole, and the Extremal Principle

Solutions · Thu Sep 24 / Fri Sep 25

Problem 1. ⋆ *(classical) Among any $n+1$ distinct elements of $\{1, \dots, 2n\}$, one divides another.*
Key idea. Every positive integer is a power of two times an odd number, and there are only n odd numbers below $2n$.

Solution. Write each chosen element a as $a = 2^k u$ with $k \geq 0$ and u odd; the odd part u is uniquely determined by a. Since $u \leq a \leq 2n$, the odd part lies in $\{1, 3, 5, \dots, 2n-1\}$, a set of exactly n elements. We have $n+1$ chosen numbers and only n possible odd parts, so by the pigeonhole principle two chosen numbers $a = 2^k u$ and $b = 2^\ell u$ have the same odd part u. They are distinct, so $k \neq \ell$; if $k < \ell$ then $b = 2^{\ell-k} a$, so a divides b, and symmetrically if $k > \ell$. □

Remark. The bound is sharp: the n numbers $n+1, n+2, \dots, 2n$ contain no pair in which one divides the other, since a proper multiple of $a \geq n+1$ is at least $2a > 2n$.

Problem 2. ⋆ *(Erdős–Szekeres) Every sequence of n^2+1 distinct reals has a strictly monotone subsequence of length $n+1$.*
Key idea. Label each term by the length of the longest increasing subsequence ending there; if all labels are small, pigeonhole forces many terms with the same label, and those terms must decrease.

Solution. Let the sequence be $x_1, x_2, \dots, x_N$ with $N = n^2+1$. For each index i let $L(i)$ be the length of the longest strictly increasing subsequence that ends at x_i; note $L(i) \geq 1$. If some $L(i) \geq n+1$ we are done, so suppose $L(i) \in \{1, 2, \dots, n\}$ for every i. We are placing n^2+1 indices into n boxes, so by the pigeonhole principle some label value is taken by at least $\lceil (n^2+1)/n \rceil = n+1$ indices; call them $i_1 < i_2 < \cdots < i_{n+1}$.

We claim $x_{i_1} > x_{i_2} > \cdots > x_{i_{n+1}}$. Indeed, suppose $x_{i_s} < x_{i_t}$ for some $s < t$. Appending x_{i_t} to a longest increasing subsequence ending at x_{i_s} produces an increasing subsequence ending at x_{i_t} of length $L(i_s)+1$, so $L(i_t) \geq L(i_s)+1$, contradicting $L(i_s) = L(i_t)$. Since the terms are distinct, $x_{i_s} > x_{i_t}$ whenever $s < t$, and the $n+1$ terms $x_{i_1}, \dots, x_{i_{n+1}}$ form a strictly decreasing subsequence. □

Remark. A second proof labels each x_i by the pair $(L(i), D(i))$, where $D(i)$ is the length of the longest decreasing subsequence ending at x_i. For $i < j$ one of $L(j) > L(i)$ or $D(j) > D(i)$ holds, so the n^2+1 pairs are distinct; they cannot all lie in $\{1, \dots, n\}^2$, which has only n^2 elements. The sharpness example is $n, n-1, \dots, 1,\ 2n, \dots, n+1,\ \dots,\ n^2, \dots, n^2-n+1$.

Problem 3. ⋆⋆ *(Putnam 2006 B2) For every set $X = \{x_1, \dots, x_n\}$ of reals there is a nonempty $S \subseteq X$ and an integer m with $\left| m + \sum_{s \in S} s \right| \leq 1/(n+1)$.*
Key idea. The fractional parts of the $n+1$ partial sums $0, s_1, \dots, s_n$ cut the circle $[0,1)$ into $n+1$ gaps of total length 1, so some gap is at most $1/(n+1)$; a gap between two partial sums is a consecutive block sum, up to an integer.

Solution. Let $s_0 = 0$ and $s_k = x_1 + \cdots + x_k$ for $1 \leq k \leq n$, and write $\{t\} = t - \lfloor t \rfloor \in [0,1)$ for the fractional part. Sort the $n+1$ numbers $\{s_0\}, \{s_1\}, \dots, \{s_n\}$ into nondecreasing order $0 = u_0 \leq u_1 \leq \cdots \leq u_n < 1$ (note $u_0 = 0$ because $\{s_0\} = 0$). The $n+1$ nonnegative quantities

$$u_1 - u_0, \quad u_2 - u_1, \quad \dots, \quad u_n - u_{n-1}, \quad 1 - u_n$$

sum to 1, so by the averaging form of the pigeonhole principle at least one of them is at most $\frac{1}{n+1}$. There are two cases.

Case 1: some $u_r - u_{r-1} \le \frac{1}{n+1}$. Then there are indices $i \ne j$ with $\{u_{r-1}, u_r\} = \{\{s_i\}, \{s_j\}\}$; relabel so that $i < j$. Then $|\{s_j\} - \{s_i\}| \le \frac{1}{n+1}$. Take $S = \{x_{i+1}, x_{i+2}, \ldots, x_j\}$, which is nonempty since $i < j$, and $m = \lfloor s_i \rfloor - \lfloor s_j \rfloor$. Then

$$m + \sum_{s \in S} s = \lfloor s_i \rfloor - \lfloor s_j \rfloor + (s_j - s_i) = \{s_j\} - \{s_i\},$$

whose absolute value is at most $\frac{1}{n+1}$.

Case 2: $1 - u_n \le \frac{1}{n+1}$. Then $u_n \ge \frac{n}{n+1} > 0$, so $u_n = \{s_k\}$ for some $k \ge 1$. Take $S = \{x_1, \ldots, x_k\}$ and $m = -\lfloor s_k \rfloor - 1$. Then $m + \sum_{s \in S} s = \{s_k\} - 1 = u_n - 1$, which lies in $[-\frac{1}{n+1}, 0)$.

In both cases $\left|m + \sum_{s \in S} s\right| \le \frac{1}{n+1}$. □

Remark. Chopping $[0, 1)$ into $n + 1$ half-open intervals of length $\frac{1}{n+1}$ and applying pigeonhole to $n + 1$ fractional parts does *not* work directly: the $n + 1$ points may occupy the $n + 1$ intervals one each. Sorting and looking at gaps (including the wrap-around gap $1 - u_n$) is the fix, and it is the same idea used to prove Dirichlet's approximation theorem.

Problem 4. ⋆⋆ *(Putnam 2012 A1) Twelve reals* $d_1, \ldots, d_{12}$ *in* $(1, 12)$*; show three of them are the sides of an acute triangle.*

Key idea. Sort the numbers; if no three consecutive ones work, the squares grow at least as fast as the Fibonacci numbers, and $F_{12} = 144$ is too big for the interval $(1, 12)$.

Solution. First a fact about triangles: if $0 < a \le b \le c$, then a, b, c are the sides of an acute triangle if and only if $c^2 < a^2 + b^2$. Indeed, that inequality gives $c^2 < (a + b)^2$, so the triangle inequality holds and a triangle exists; the largest angle is opposite c, and by the law of cosines it is acute exactly when $c^2 < a^2 + b^2$; the other two angles are then automatically acute.

Now relabel so that $d_1 \le d_2 \le \cdots \le d_{12}$ (the extremal step: sorting). Suppose, for contradiction, that no three of the d_i form an acute triangle. Then in particular, for each $1 \le i \le 10$, the sorted triple $d_i \le d_{i+1} \le d_{i+2}$ fails the test above, so

$$d_{i+2}^2 \ge d_i^2 + d_{i+1}^2 \qquad (1 \le i \le 10).$$

Let $F_1 = F_2 = 1$ and $F_{k+2} = F_{k+1} + F_k$ be the Fibonacci numbers. We claim $d_k^2 \ge F_k\, d_1^2$ for $1 \le k \le 12$, by strong induction on k: for $k = 1$ this is an equality, for $k = 2$ it says $d_2^2 \ge d_1^2$, true by the ordering, and for $k \ge 3$,

$$d_k^2 \ge d_{k-1}^2 + d_{k-2}^2 \ge (F_{k-1} + F_{k-2})\, d_1^2 = F_k\, d_1^2.$$

With $k = 12$ we get $d_{12}^2 \ge F_{12}\, d_1^2 = 144\, d_1^2 > 144$, because $d_1 > 1$. Hence $d_{12} > 12$, contradicting $d_{12} < 12$. So some three of the numbers are the sides of an acute triangle. □

Remark. The number 12 is exactly right: with $d_k = \sqrt{F_k}$ scaled to sit just inside $(1, 12)$ one gets eleven numbers, no three of which form an acute triangle, but a twelfth cannot be added. The problem is a disguised "$n + 1$ objects" statement, and sorting is what reveals it.

Problem 5. ⋆⋆⋆ *(Putnam 2016 B3)* S *is a finite set of points in the plane, every triangle with vertices in* S *has area at most* 1*. Show some triangle of area* 4 *covers* S*.*

Key idea. Take the triangle ABC of maximum area with vertices in S; the triangle twice its size having A, B, C as midpoints of its sides has area $4[ABC] \le 4$, and every point of S is trapped inside it by maximality.

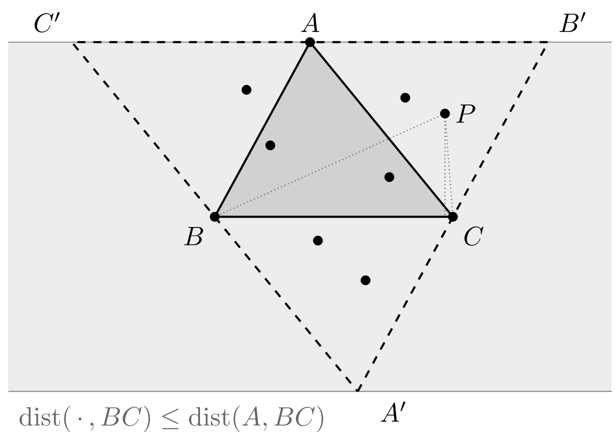
Solution. Write $[XYZ]$ for the area of triangle XYZ. If S is empty, has at most two points, or lies on a line, then S is contained in a segment, which is contained in a triangle of area 4 (put the segment along one side of a triangle with an appropriate height). So assume S contains three non-collinear points.

Since S is finite, there are finitely many triples in S, so we may choose $A, B, C \in S$ with $[ABC]$ maximal among all triangles with vertices in S; by the previous paragraph $[ABC] > 0$, and by hypothesis $[ABC] \le 1$. This is the extremal principle. Now define

$$A' = B + C - A, \qquad B' = C + A - B, \qquad C' = A + B - C$$

(vector notation). Then A is the midpoint of $B'C'$, B the midpoint of $C'A'$, C the midpoint of $A'B'$, and $C' - B' = 2(B - C)$, so the side $B'C'$ is parallel to BC and twice as long; likewise for the other sides. Thus $A'B'C'$ is similar to ABC with ratio 2 and $[A'B'C'] = 4[ABC] \le 4$.

We show every $P \in S$ lies in the closed triangle $A'B'C'$. By maximality, $[PBC] \le [ABC]$, that is, $\operatorname{dist}(P, BC) \le \operatorname{dist}(A, BC)$. The set of points at distance at most $\operatorname{dist}(A, BC)$ from the line BC is the closed strip between the two lines parallel to BC at that distance, and one of these two lines is $B'C'$ (it passes through A and is parallel to BC). In particular P lies in the closed half-plane bounded by $B'C'$ that contains B and C. By the same argument with A, B, C permuted, P lies in the closed half-plane bounded by $C'A'$ containing C, A, and in the closed half-plane bounded by $A'B'$ containing A, B. The intersection of these three closed half-planes is exactly the closed triangle $A'B'C'$, so $P \in A'B'C'$.



The maximal triangle ABC (shaded) and the triangle $A'B'C'$ with A, B, C as side midpoints (dashed). Maximality of $[ABC]$ says no point P of S is farther from line BC than A is, so S lies in the grey strip, and hence on the BC side of line $B'C'$. Repeating for the other two sides traps S in $A'B'C'$.

Finally, if $[A'B'C'] < 4$, apply the homothety centered at A' with ratio $\lambda = \sqrt{4/[A'B'C']} \ge 1$. The image is a triangle of area exactly 4, and it still contains $A'B'C'$: for any point X of the closed triangle, the point $Y = A' + (X - A')/\lambda$ lies on the segment $A'X$, hence in the triangle by convexity, and the homothety sends Y to X. So the image covers S. $\square$

Remark. Note where finiteness was used: only to guarantee that a triangle of maximal area exists. That is the one thing the extremal principle needs, and the one thing a grader will look for. The three half-planes are just the three conditions "no point of S is farther from BC than A is," and their intersection happens to be a triangle four times the size of ABC; once you draw that picture the proof writes itself.

Problem 6. ⋆⋆⋆⋆ *(Putnam 2002 A5)* $a_0 = 1$, $a_{2n+1} = a_n$, $a_{2n+2} = a_n + a_{n+1}$ *for* $n \geq 0$. *Show every positive rational appears as* a_{n-1}/a_n *for some* $n \geq 1$.

Key idea. Prove the stronger statement that every pair of coprime positive integers (r, s) occurs as consecutive terms (a_n, a_{n+1}), by strong induction on $r + s$; the recursion is the Euclidean algorithm run backwards.

Solution. Every positive rational is r/s with r, s coprime positive integers, and $r/s = a_n/a_{n+1} = a_{(n+1)-1}/a_{n+1}$ as soon as $a_n = r$ and $a_{n+1} = s$ for some $n \geq 0$. So it suffices to prove:

Claim. For every pair (r, s) of coprime positive integers there is an $n \geq 0$ with $a_n = r$ and $a_{n+1} = s$.

We prove the claim by strong induction on $r + s \geq 2$. If $r + s = 2$ then $r = s = 1$, and $n = 0$ works since $a_0 = 1$ and $a_1 = a_{2\cdot 0+1} = a_0 = 1$.

Now let $r+s > 2$ and assume the claim for all coprime pairs with smaller sum. Since $\gcd(r, s) = 1$ and $(r, s) \neq (1, 1)$, we have $r \neq s$.

Case $r > s$. The pair $(r - s, s)$ consists of positive integers with $\gcd(r - s, s) = \gcd(r, s) = 1$ and smaller sum r, so by the inductive hypothesis there is n with $a_n = r - s$ and $a_{n+1} = s$. Then by the recursion,

$$a_{2n+2} = a_n + a_{n+1} = r, \qquad a_{2n+3} = a_{2(n+1)+1} = a_{n+1} = s,$$

so the consecutive pair (a_{2n+2}, a_{2n+3}) equals (r, s).

Case $r < s$. The pair $(r, s - r)$ is coprime with sum $s < r + s$, so there is n with $a_n = r$ and $a_{n+1} = s - r$. Then

$$a_{2n+1} = a_n = r, \qquad a_{2n+2} = a_n + a_{n+1} = s,$$

so $(a_{2n+1}, a_{2n+2}) = (r, s)$.

This completes the induction and the proof. □

Remark. Reading the proof forwards: the pair (a_n, a_{n+1}) generates the two children $(a_{2n+1}, a_{2n+2}) = (a_n, a_n + a_{n+1})$ and $(a_{2n+2}, a_{2n+3}) = (a_n + a_{n+1}, a_{n+1})$. Starting from $(1, 1)$ this is the Calkin–Wilf tree, and one can show with a second induction that each coprime pair occurs exactly once, so $n \mapsto a_{n-1}/a_n$ is a bijection from the positive integers to the positive rationals. The induction parameter $r + s$ is the whole trick; inducting on n leads nowhere.