

Week 1: Induction, Pigeonhole, and the Extremal Principle

Handout · Thu Sep 24 / Fri Sep 25

Pigeonhole: N objects in n boxes put at least $\lceil N/n \rceil$ in one box; the skill is choosing the boxes (residues, odd parts, partial sums, short intervals). **Extremal principle:** in a finite configuration pick the largest, smallest, or closest object and exploit that nothing beats it; sorting is the same idea. **Induction:** state $P(n)$ exactly, choose the parameter carefully (it need not be the obvious n), use strong induction freely, and strengthen the statement when the step will not close. Pigeonhole and extremal arguments usually run by contradiction, so start with “suppose not”; induction is the constructive counterpart.

The exam: Saturday, December 5, 2026. New this year: four 90-minute sessions with breaks (rather than two 3-hour sessions); 12 problems A1–A6, B1–B6, 10 points each, graded almost entirely 0–2 or 8–10. The national median is usually 0 or 1; one fully written solution is a real achievement.

Problem 1. ★ (classical) Let $n \geq 1$. Show that among any $n + 1$ distinct elements of $\{1, 2, \dots, 2n\}$, one of them divides another.

Problem 2. ★ (Erdős–Szekeres) Let $n \geq 1$. Show that every sequence of $n^2 + 1$ distinct real numbers contains a monotone (strictly increasing or strictly decreasing) subsequence of length $n + 1$.

Problem 3. ★★ (Putnam 2006 B2) Prove that, for every set $X = \{x_1, x_2, \dots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left| m + \sum_{s \in S} s \right| \leq \frac{1}{n+1}.$$

Problem 4. ★★ (Putnam 2012 A1) Let $d_1, d_2, \dots, d_{12}$ be real numbers in the open interval $(1, 12)$. Show that there exist distinct indices i, j, k such that d_i, d_j, d_k are the side lengths of an acute triangle.

Problem 5. ★★★ (Putnam 2016 B3) Suppose that S is a finite set of points in the plane such that the area of triangle ABC is at most 1 whenever A, B , and C are in S . Show that there exists a triangle of area 4 that (together with its interior) covers the set S .

Problem 6. ★★★★★ (Putnam 2002 A5) Define a sequence by $a_0 = 1$, together with the rules $a_{2n+1} = a_n$ and $a_{2n+2} = a_n + a_{n+1}$ for each integer $n \geq 0$. Prove that every positive rational number appears in the set

$$\left\{ \frac{a_{n-1}}{a_n} : n \geq 1 \right\} = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \dots \right\}.$$

Problems 1–4 are the in-session targets; 5–6 are take-home. Solutions will be posted after Friday’s session.