POLY-BERNOULLI NUMBERS & MATCHSTICK GAMES ON CYLINDERS

REED COLLEGE COLLABORATIVE MATHEMATICS RESEARCH GROUP 2024

AMELIE EL MAHMOUD, CAOILAINN KIRKPATRICK, KYLE ORMSBY, ANGÉLICA OSORNO, DALE SCHANDELMEIER-LYNCH, RILEY SHAHAR, LIXING YI, AVERY YOUNG, SARON ZHU

JMM — 2025 JANUARY 9

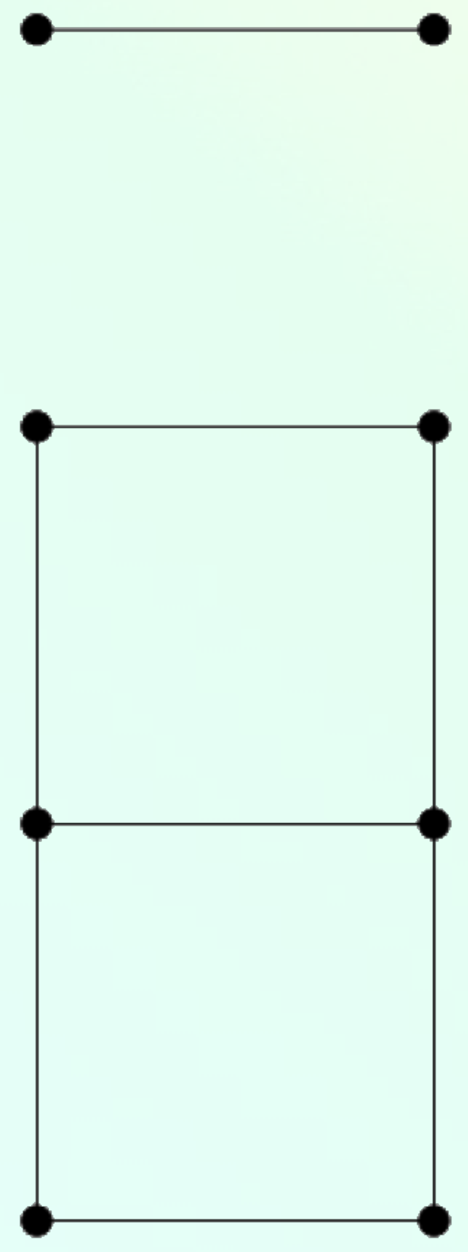
MATCHSTICK GAMES

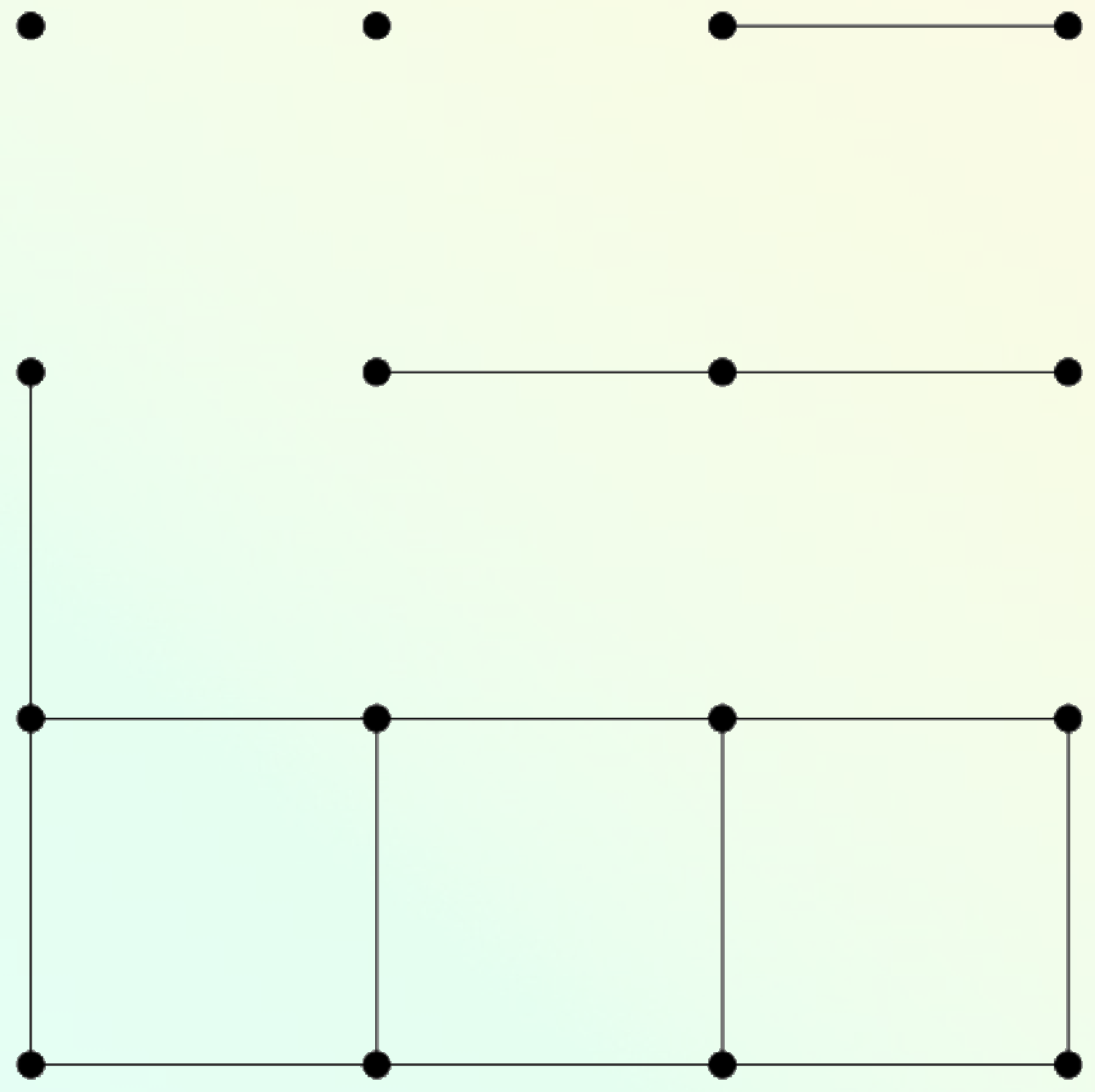
SATURATED TRANSFER SYSTEMS ON MODULAR LATTICES

Write $[m] = \{0 < 1 < \cdots < m\}$ so that $[m] \times [n]$ is a **rectangular grid** poset.

MATCHSTICK GAME RULES:

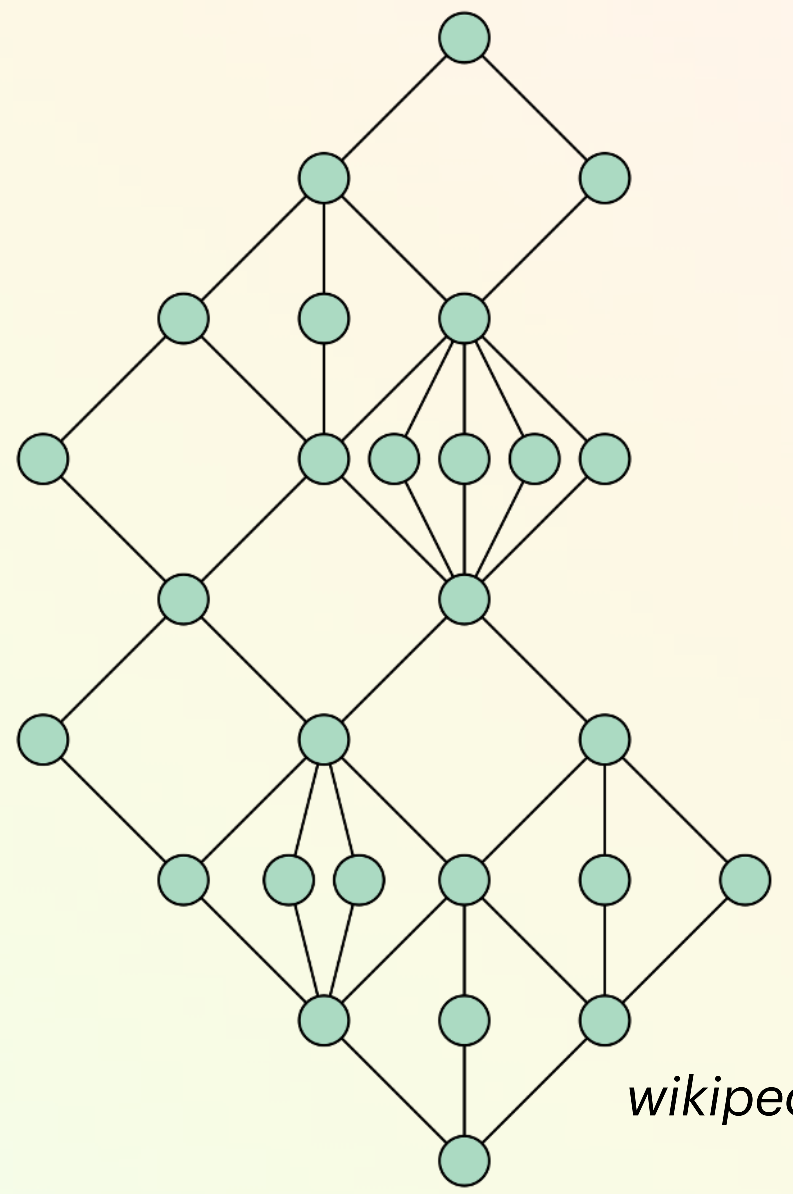
- Vertical stick implies all sticks to its left
- Horizontal stick implies all sticks below it
- $3 \implies 4$ in unit boxes





MODULAR LATTICES

KEY STRUCTURAL PROPERTY OF SUBGROUP LATTICES OF ABELIAN GROUPS


wikipedia

A lattice P is **modular** when $x \leq y$ implies $x \vee (z \wedge y) = (x \vee z) \wedge y$.

Equivalently, $[x \wedge y, y] \cong [x, x \vee y]$ for all x, y (**diamond isomorphism**).

MATCHSTICK GAME RULES:

- Q a subset of covering relations of P
- $x \; Q \; (x \vee y) \implies (x \wedge y) \; Q \; y$
- $3 \implies 4$ in covering diamonds

THEOREM. A lattice is modular if and if it contains no pentagonal sublattice.

WHY MATCHSTICK GAMES?

Matchstick games on modular lattice P enumerate

- saturated transfer systems
- submonoids of $(P, \vee)$
- max-closed relations (Knuth)
- interior operators
- coreflective factorization systems
- cofibrant model structures

(See Tien Chih's talk!)

COUNTING GAMES

THEOREM (Hafeez-Marcus-O-Osorno '22). The number of legal matchstick games on $[m] \times [n]$ is

$$\mathrm{games}([m] \times [n]) = \sum_{j=2}^{m+2} (-1)^{m-j} \left\{ {m+1 \atop j-1} \right\} \frac{j!}{2} j^n$$

for $\left\{ {r \atop s} \right\}$ the Stirling number of the second kind counting s-block partitions of an r-element set, and these numbers satisfy the recurrence

$$\mathrm{games}([m] \times [n+1]) = \mathrm{games}([m] \times [n]) + \sum_{j=0}^{m} \binom{m+1}{j} \mathrm{games}([j] \times [n])\,.$$

POLY-BERNOULLI NUMBERS

BERNOULLI NUMBERS + POLYLOGARITHMS = COMBINATORICS

COROLLARY. Matchstick games on $[m] \times [n]$ are seminumerous with poly-Bernoulli numbers: $2\,\mathrm{games}([m] \times [n]) = B_{m+1,n+1}$.

The **poly-Bernoulli numbers** $B_n^{(s)}$ [Kaneko 1997] are defined by

$$\sum_{n \geq 0} B_n^{(s)} \frac{z^n}{n!} = \frac{1}{1 - e^{-z}} \sum_{k \geq 1} \frac{(1 - e^{-z})^k}{k^s}.$$

Then $B_n^{(1)}$ is the classical Bernoulli numbers, and $B_{m,n} := B_n^{(-m)}$ is a positive integer for $m, n \in \mathbb{N}$.

POLY-BERNOULLI NUMBERS

BERNOULLI NUMBERS + POLYLOGARITHMS = COMBINATORICS

THEOREM [Kaneko; see Knuth 2024]. The (re-indexed) pB numbers satisfy

$$B_{m,n} = \sum_{k\geq 0} (k!)^2 \left\{ {m+1 \atop k+1} \right\} \left\{ {n+1 \atop k+1} \right\}$$

and their exponential generating function is

$$G(x,y) = \sum_{m,n\geq 0} B_{m,n} \frac{x^m y^n}{m!n!} = \frac{e^{x+y}}{e^x + e^y - e^{x+y}}.$$

CAN WE GENERALIZE THESE FORMULÆ SO THEY APPLY TO OTHER MODULAR LATTICES?

WE WILL FOCUS ON LATTICES OF THE FORM $P \times [n]$ FOR P MODULAR.

TRANSFER MATRIX METHOD

BUILDING $\mathrm{games}(P \times [n])$ **ONE LAYER AT A TIME**

If G is a weighted directed graph with adjacency matrix $A(G)$, then

$$(A(G)^n)_{u,v} = \sum_{\substack{\text{length } n \text{ walks} \\ v_0 v_1 \cdots v_n \\ u = v_0,\ v = v_n}} \prod_{i=0}^{n-1} \mathrm{weight}(v_i v_{i+1})\,.$$

Define $G(P)$ to have vertex set $\mathrm{games}(P)$ and a directed edge QQ' with weight the number of matchstick games on $P \times [1]$ with Q on 'bottom' and Q' on 'top'.

For $A(P) := A(G(P))$, we have

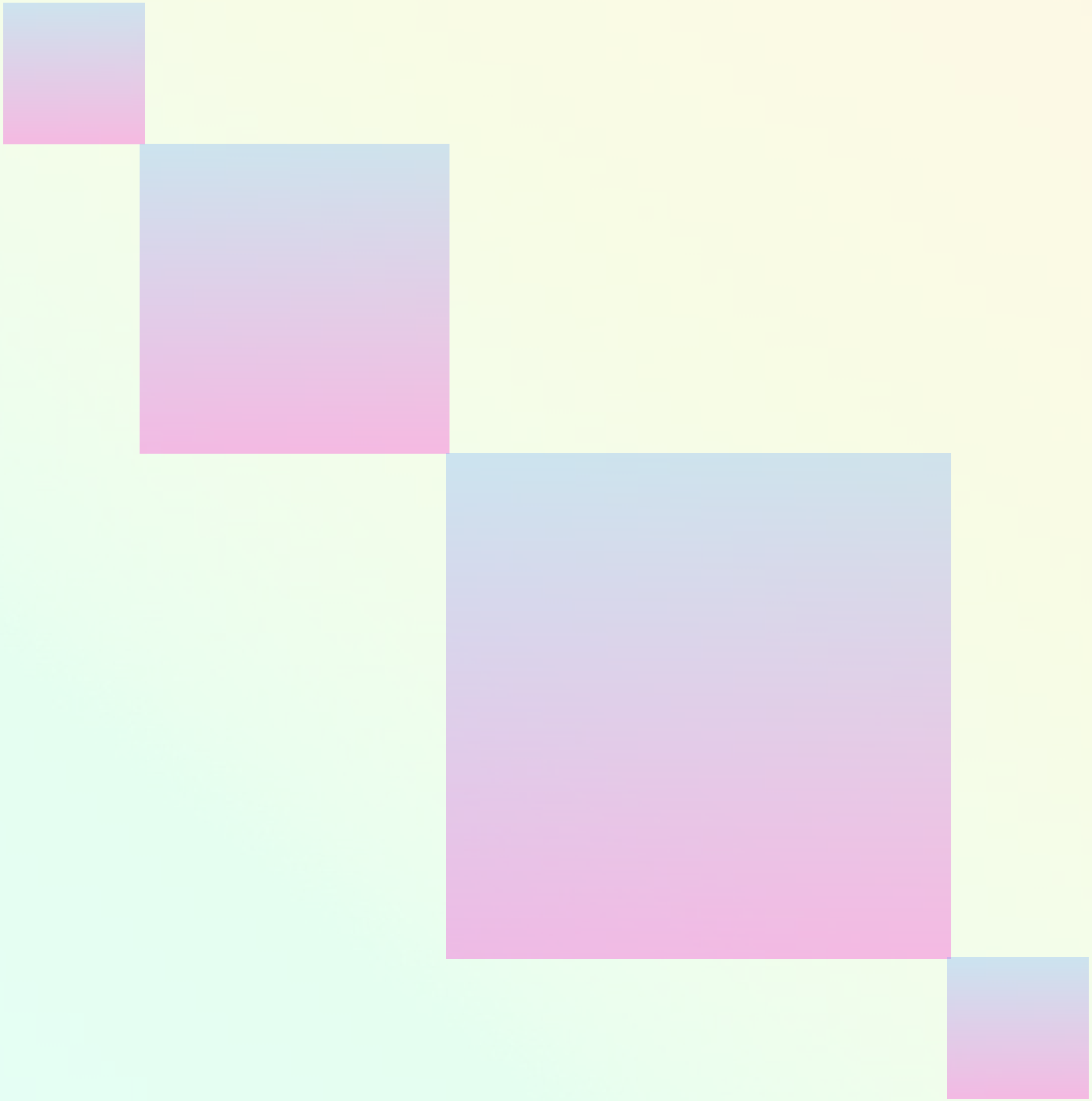
$$\mathrm{games}(P \times [n]) = \sum_{Q,Q' \in \mathrm{games}(P)} (A(P)^n)_{Q,Q'}\,.$$

EXAMPLE

SPECIALIZING TO $P = [1] \times [1]$

DIAGONALIZABILITY

LEMMA [CMRG ’24]. For any finite modular lattice P there is a linear extension of $\mathrm{games}(P)$ such that $A(P)$ is upper triangular with equal diagonal entries contiguous and each contiguous block diagonal. It follows that $A(P)$ is diagonalizable.



FORMULÆ & ASYMPTOTICS

THEOREM [CMRG '24]. For each finite modular lattice P, there are rational numbers b_i and positive integers λ_i, $1 \leq i \leq m$, such that

$$\#\text{games}(P \times [n]) = \sum_{i=1}^{m} b_i \lambda_i^n .$$

COROLLARY [CMRG '24]. The largest eigenvalue of $A(P)$ equals $\text{ac}(P)$, the number of antichains of P, and thus

$$\#\text{games}(P \times [n]) = \Theta(\text{ac}(P)^n) .$$

EXAMPLE

SPECIALIZING TO $[1] \times [1] \times [n]$

Set $P = [1] \times [1]$. The number of matchstick games on $P \times [n] = [1] \times [1] \times [n]$ is

$$\#\text{games}([1] \times [1] \times [n]) = \frac{35}{2} \cdot 6^n - 12 \cdot 4^n + 3^n + \frac{1}{2} \cdot 2^n \,.$$

(An equivalent formula was discovered independently by Filip Stappers in the context of “max-closed relations”.)

FORMULÆ

[1]^{3}

[2]^{*5}

GENERATING FUNCTIONS

EXPONENTIAL GENERATING FUNCTION FOR HIGHER-DIMENSIONAL GRIDS

Set $G(x_1, \ldots, x_k) = \sum_{n_1, \ldots, n_k \geq 0} 2\#\text{games}([n_1 - 1] \times \cdots \times [n_k - 1]) \frac{x_1^{n_1} \cdots x_k^{n_k}}{n_1! \cdots n_k!}$.

By [Kaneko '97], $G(x, y, 0, \ldots, 0) = \frac{e^{x+y}}{e^x + e^y - e^{x+y}}$.

CONJECTURE [CMRG '24]. The exponential generating function $G(x_1, \ldots, x_k)$ is a rational function in the variables $e^{x_1}, \ldots, e^{x_k}$.

BIBLIOGRAPHY

- Hafeez-Marcus-Ormsby-Osorno (2022), *Saturated and linear isometric transfer systems for cyclic groups of order $p^m q^n$*, Topology and its Applications, **317**: 1-20
- Kaneko (1997), *Poly-Bernoulli numbers*, Journal de Théorie des Nombres de Bordeaux, **9** (1): 221–228
- Knuth (2024), *Parades and poly-Bernoulli numbers*
- Stappers (2024), OEIS A370966

ACKNOWLEDGMENTS

NSF DMS-2204365, DMS-1916439

American Mathematical Society

Reed College