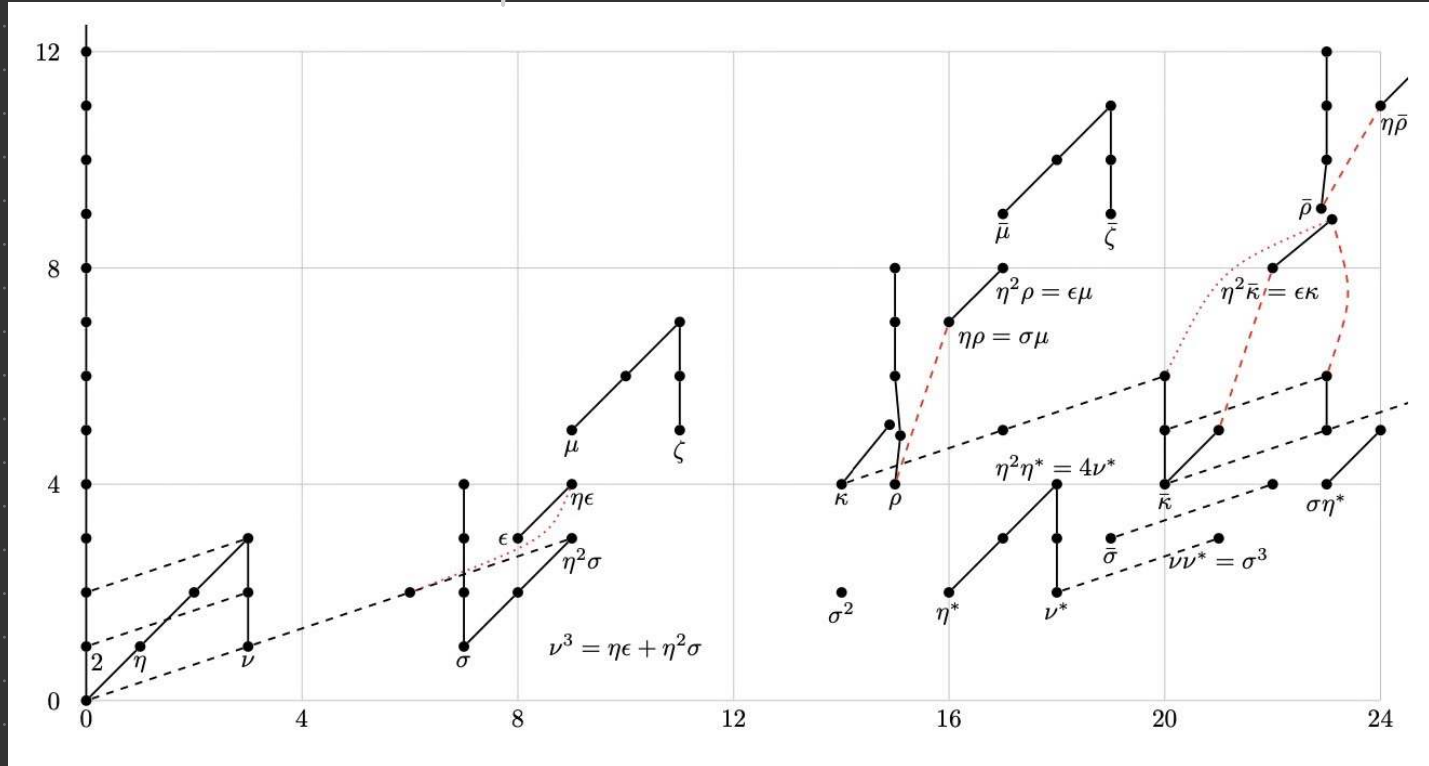


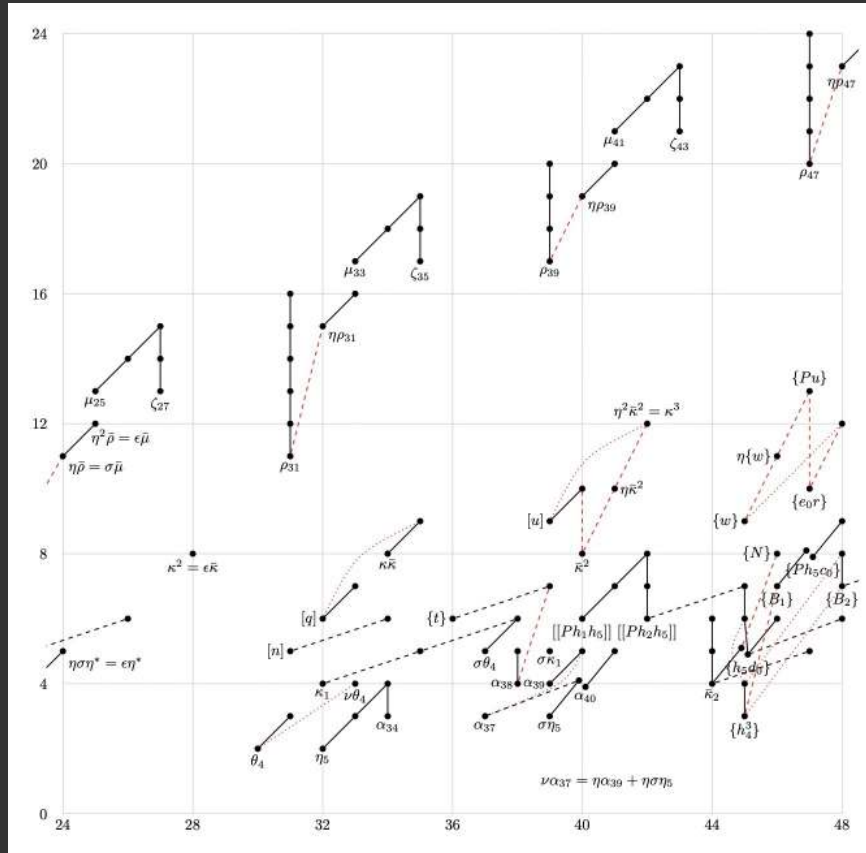
Some homotopy groups of S

18. IV. 23

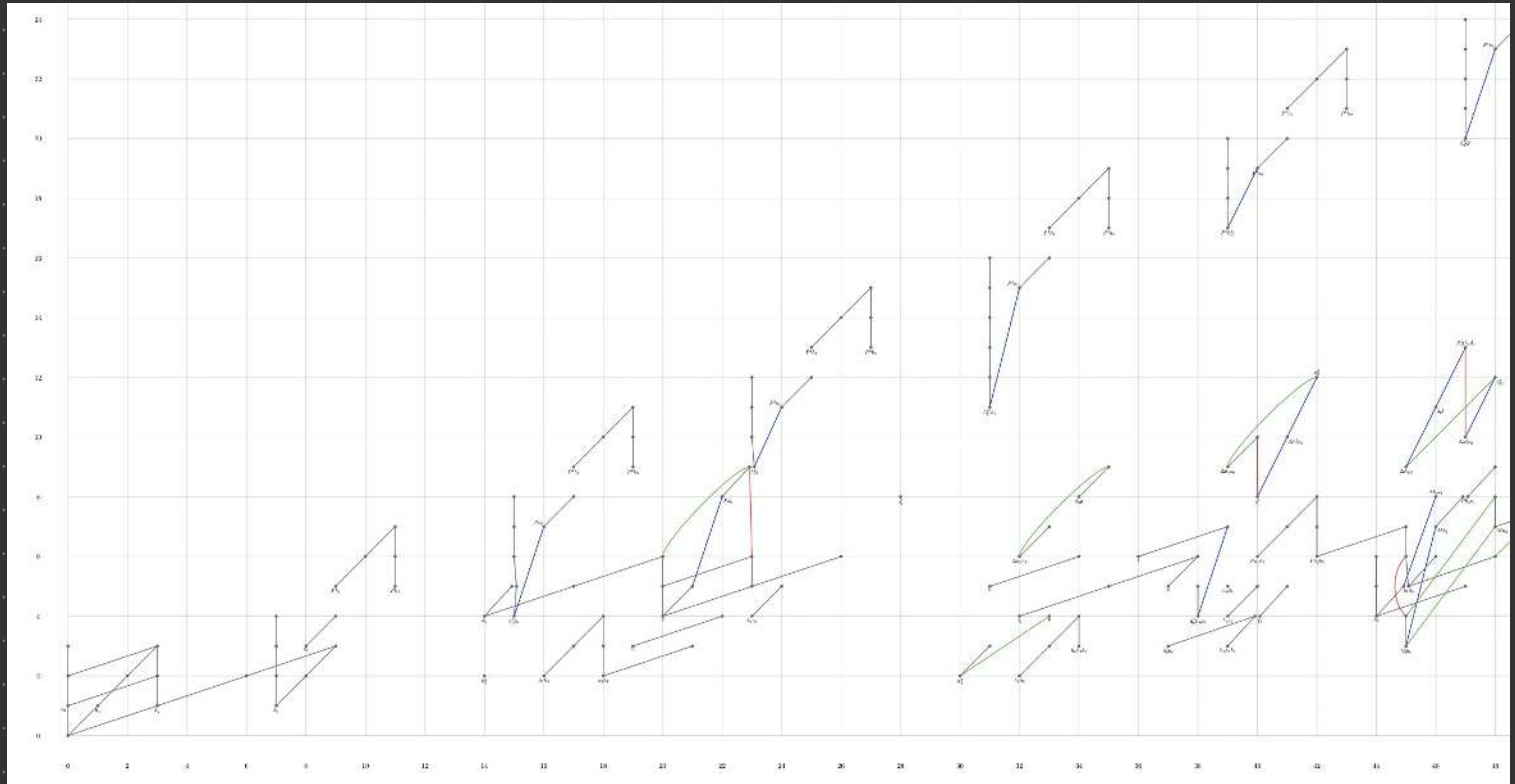
I. Understand this picture:



II. And this one :



III. So that we ultimately understand



and its ring structure.

0. History



L.E.J. Brouwer:
1911
degree thm



Heinz Hopf:

1927

$$\pi_n(S^m) = \mathbb{Z}$$

1931

η, ν, σ essential



Hans Freudenthal:
1938
stable stems
 $\pi_* S = \mathbb{Z}/2$



Lev Pontryagin,
George V. Whitehead:
1950
 $\pi_2 S = \mathbb{Z}/2$



Hiroshi Toda,
JP Serre
1952-58
 $\pi_n S, 3 \leq n \leq 13$
(Toda 1962: $n \leq 19$)



Mamoru Mimura
1963 w/ Toda $\pi_{20} S$
1965
 $\pi_{21} S, \pi_{22} S$ (EK $\neq 0$)



J. Frank Adams
1958

Adams spectral sequence



J. Peter May
1965

$n \leq 28$ except $n=23$

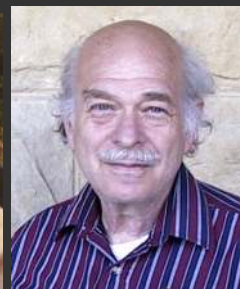


MT

MM

Mark Mahowald,
Martin Tangora
1967

$n \leq 37$, $n \in \{39, 42, 43, 44\}$
but...



Differentials
corrected w/
Michael Barratt
(1970) and by
R. James Milgram
(1972) $\Rightarrow n \leq 44$
except 37, 38



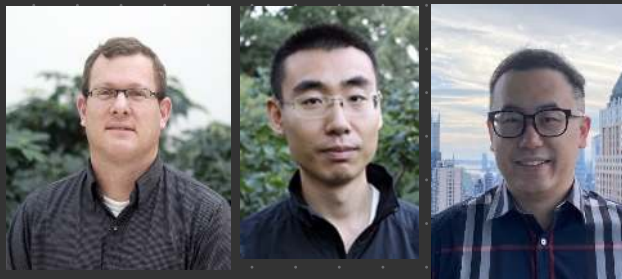
Bob Bruner
1984

New Adams differential
 $\Rightarrow \pi_{37} S, \pi_{38} S$



Stanley Kochman
1990

$n \leq 53$ except 51, $58 \leq n \leq 60$;
 $n=55$ corrected with Mahowald
(1995)

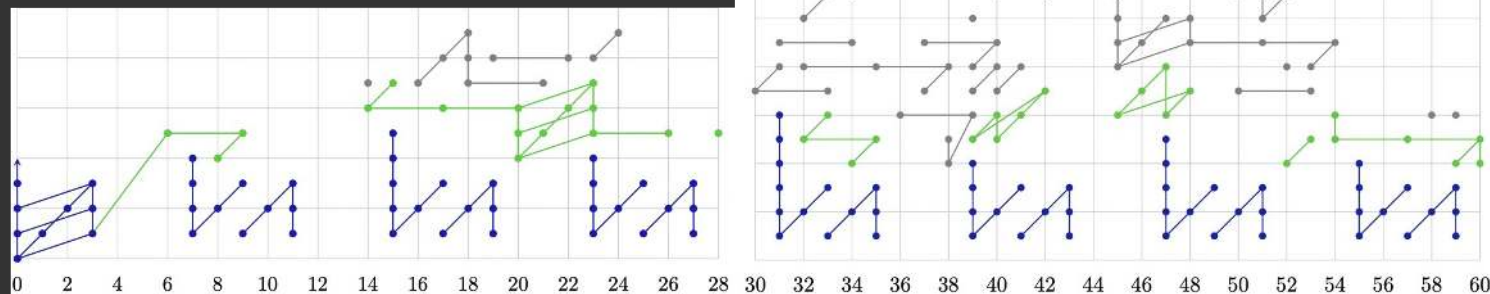


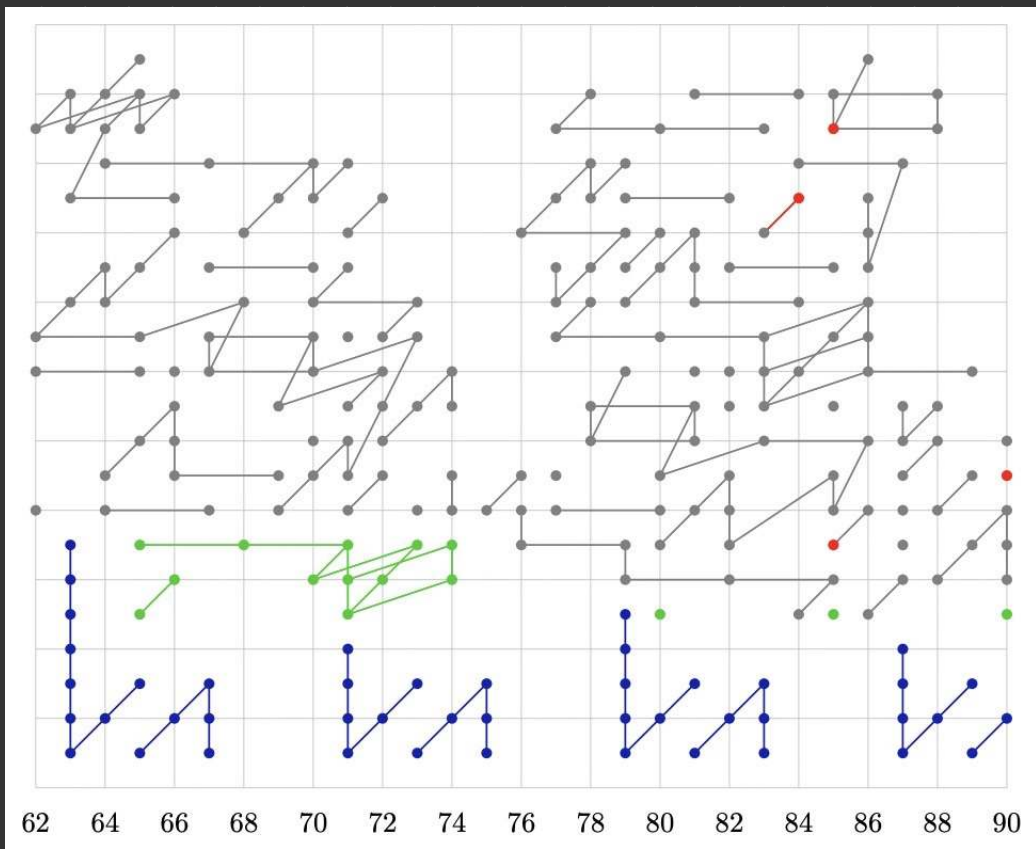
Dan Isaksen, Guozhen Wang, Zhouli Xu

2015
new Adams diff's
for 51, 56 stems

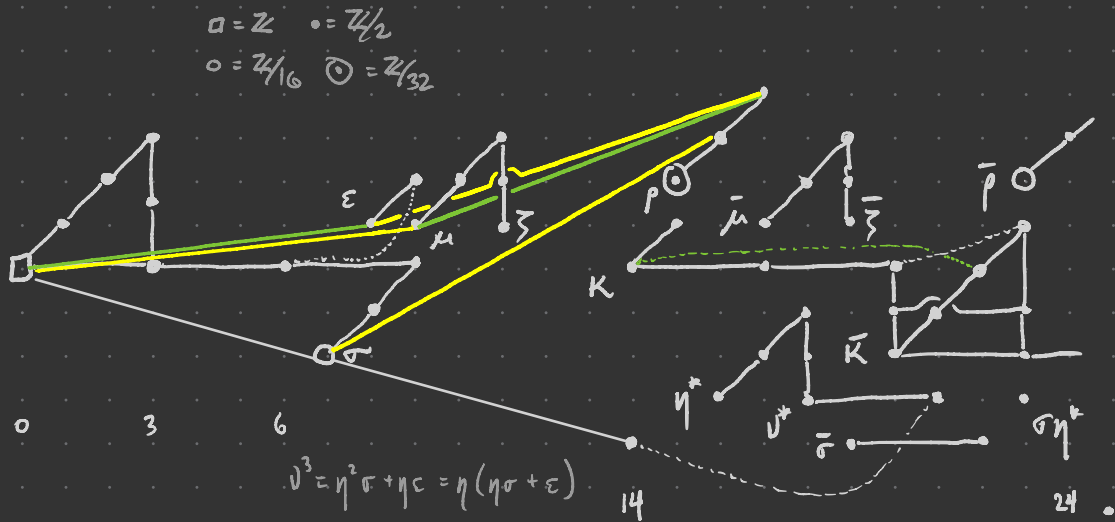
2017-18
 $\pi_{51}S, \pi_{61}S$
so $n \leq 61$

All three, arXiv 2020: $n \leq 90$ except for some
very precise uncertainties in $n = 84, 85, 90$





I. Thm The ring structure of $\pi_* S$ for $* \leq 24$ is given by



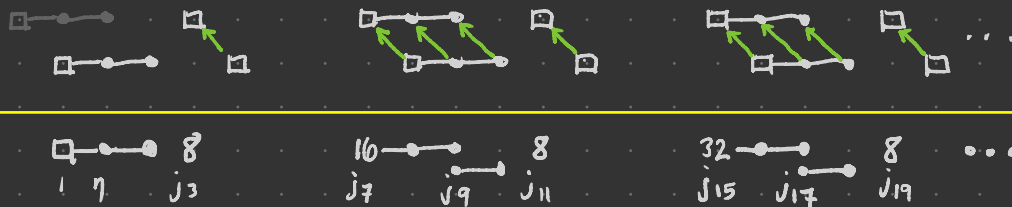
Method (a) We already know $E_5(5) = E_\infty(5)$ for the Adams spectral sequence in this range.

(b) We know $\pi_* j$ and $\pi_* \text{tmf}$ in this range.

(c) Consider the ring maps $\pi_* j \xrightarrow{e} \pi_* S$ to deduce product structure on $\pi_* S$.

$\downarrow$
 $\pi_* tmf$

Recall Fiber sequence $j \rightarrow ko \xrightarrow{\eta^{3-1}} bspin$



LEMMA 11.46. The map $e: S \rightarrow j$ is (at least) 2-connected, and for $n \geq 2$

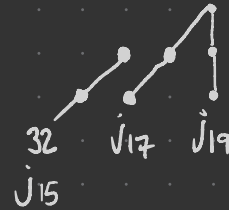
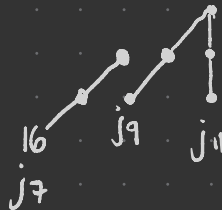
$$\pi_n(j) = \begin{cases} \mathbb{Z}_2/(16k)\{j_{8k-1}\} & \text{for } n = 8k - 1, \\ \mathbb{Z}/2\{\eta j_{8k-1}\} & \text{for } n = 8k, \\ \mathbb{Z}/2\{\eta^2 j_{8k-1}\} \oplus \mathbb{Z}/2\{j_{8k+1}\} & \text{for } n = 8k + 1, \\ \mathbb{Z}/2\{\eta j_{8k+1}\} & \text{for } n = 8k + 2, \\ \mathbb{Z}/8\{j_{8k+3}\} & \text{for } n = 8k + 3, \\ 0 & \text{otherwise,} \end{cases}$$

with $\nu j_{8k-1} = 0$ and $\eta^2 j_{8k+1} = 4j_{8k+3}$.

Furthermore, $j \xleftarrow{e} s \xrightarrow{o} k_0 \xrightarrow{\eta^3_{-1}} bspin$ induces

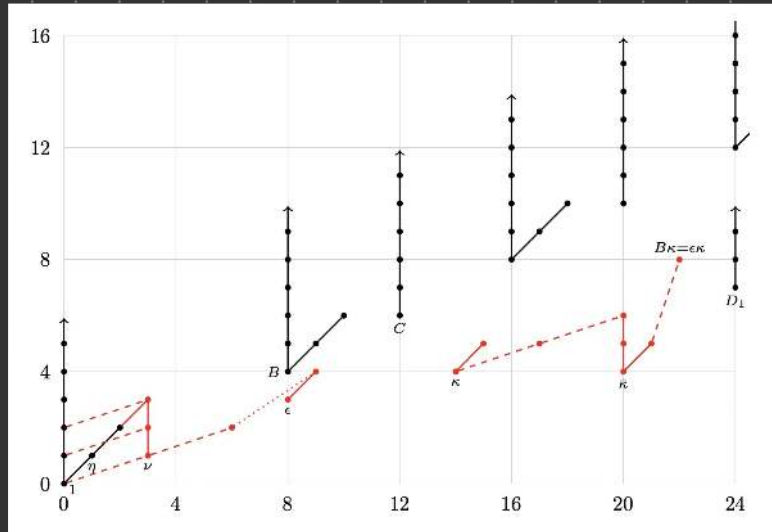
$e: \pi_+ s \rightarrow \pi_+ j$ surjective with additive section, and:

PROPOSITION 11.49 (cf. [8, Prop. 12.14 and Ex. 12.15]). The products of the j_n are given as follows: $j_{8k-1} \cdot j_{8l+1} = j_{8k+1} \cdot j_{8l-1} = \eta j_{8(k+l)-1}$, $j_{8k+1} \cdot j_{8l+1} = \eta j_{8(k+l)+1}$, and the remaining products are zero.



Slogan: $j_7 j_1 = j_1 j_7 = \eta j_7$, $j_1^2 = \eta j_1$ (indices mod 8)

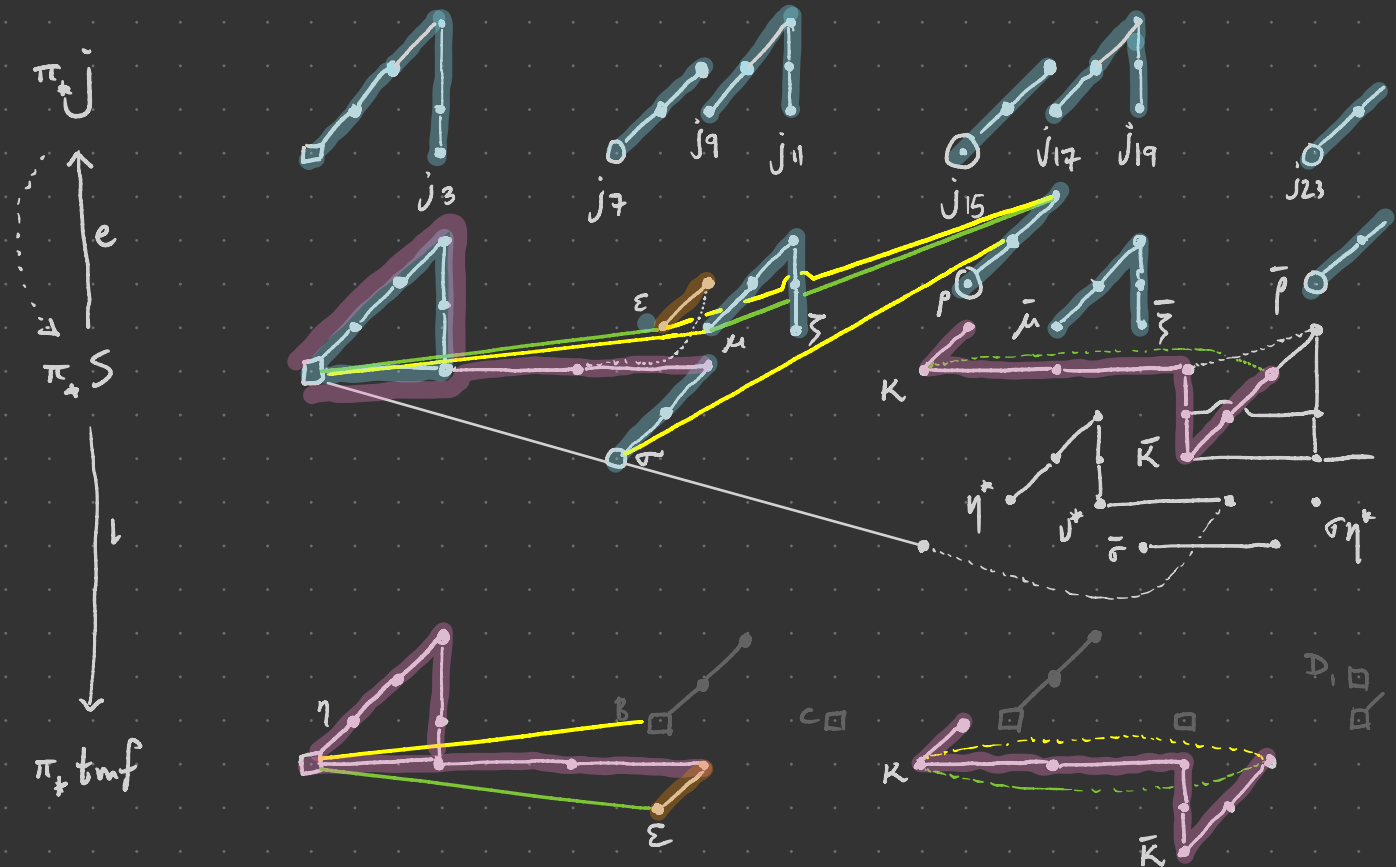
Also recall $\pi_* \text{tmf}$ ring structure:



In "constellation form", this becomes:



Now compare $\pi_* j \xleftarrow{e} \pi_* S \longrightarrow \pi_* tmf$:

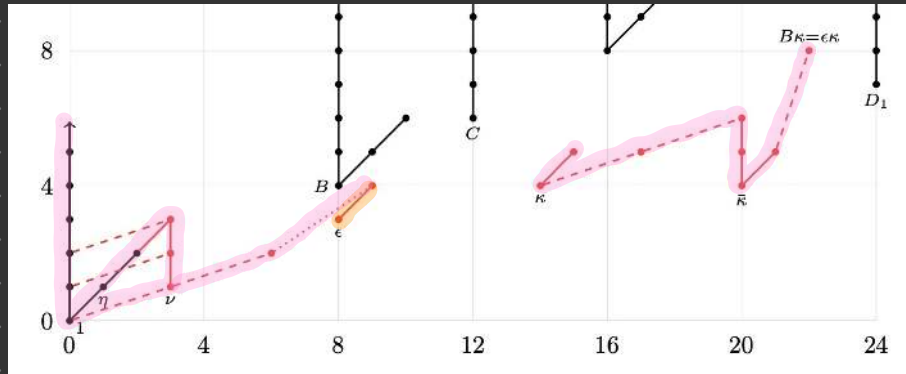
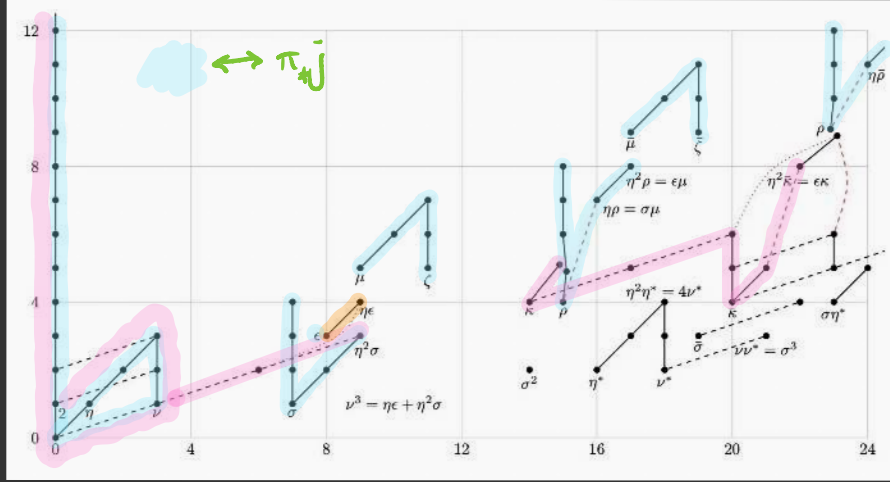


In terms of Adams charts:

$\pi_* S$

$\hookrightarrow$

$\pi_* tmf$



$$\pi_* S = \mathbb{Z}/2\{\mu, \eta\epsilon, \eta^2\sigma\}$$

$$\text{with } e(\mu) = j_9$$

$$e(\eta^2\sigma) = \eta^2 j_7$$

By $E_2(K)$ ring structure,

$$v^3 = \eta^2\sigma + \text{h.o.t.}$$

Thus $v^3 = x\mu + y\eta\epsilon + \eta^2\sigma$
for some $x, y \in \{0, 1\}$

$$e(v^3) = v e(v^2) = 0 \text{ so}$$

$$0 = e(x\mu + y\eta\epsilon + \eta^2\sigma)$$

$$= x j_9 + y \eta e(\epsilon) + \eta^2 j_7$$

$$\Rightarrow x=0, y=1,$$

$$\eta e(\epsilon) = \eta^2 j_7$$

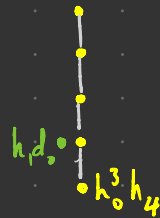
$$\text{Thus } v^3 = \eta\epsilon + \eta^2\sigma$$



$\pi_{15}S$: For $* \leq 14$, classical arguments with j suffice for ring structure. tmf starts to shine at $\pi_{15}S$.

- $E_\infty(S) \Big|_{t-s=15} = \mathbb{F}_2\{h_0^k h_4 \mid 3 \leq k \leq 7\} \oplus \mathbb{F}_2\{h, d_0\}$

- $h_0^3 h_4$ detects ρ , $h, d_0 = \eta\kappa$ with $2 \cdot \eta\kappa = 0$



$$\Rightarrow \pi_{15}S = \mathbb{Z}/2\{\eta\kappa\} \oplus \mathbb{Z}/32\{\rho\} \text{ with } \rho \text{ determined modulo } (\eta\kappa, 2\rho)$$

- Fix a choice of ρ with the following equivalent conditions:

- $\varepsilon \cdot \rho = 0 \in \pi_{23}S$

- $\iota(\rho) = 0 \in \pi_{15}tmf = \mathbb{Z}/2\{\eta\kappa\}$

} determine ρ up to an odd multiple

(Argument with $tmf/s \Rightarrow (\iota(\rho) = 0 \Rightarrow \varepsilon\rho = 0)$)

($\eta\kappa \neq 0 \in \pi_{23}S \Rightarrow (\varepsilon\rho = 0 \Rightarrow \iota(\rho) = 0)$)

- $\rho := J(\text{gen } \pi_{15}SO)$

- $e(p) \doteq j_{15}$ generates $\pi_{15} j = \mathbb{Z}/32 \{j_{15}\}$
- Now check ring structure: $\sigma\varepsilon = 0$
 - $\eta\sigma^2 = 0$ by quadratic construction on $\sigma: S^7 \rightarrow S^4$.
 - $e(\sigma\varepsilon) = \eta\sigma j_7 = 0 \in \pi_{15}(j) \Rightarrow \sigma\varepsilon \in \ker(e) = \{0, \eta\kappa\}$.
 - Have $\iota(\sigma) = 0$ and $\iota(\eta\kappa) \neq 0$, so $\sigma\varepsilon \neq \eta\kappa$. $\square$

Fun with $\varepsilon\kappa \in \pi_{22}S$:

- $\nu^2\kappa = 4\bar{\kappa} \Rightarrow \nu^3\kappa = 4\nu\bar{\kappa}$ detected by $h_1\text{Pd}_0$
- $\nu^3 = \eta\varepsilon + \eta^2\sigma$ & $\eta^2\sigma\kappa = 0 \Rightarrow \nu^3\kappa = \eta\varepsilon\kappa = 4\nu\bar{\kappa} \neq 0$
- In particular, $\varepsilon\kappa \neq 0$ detected by Pd_0 , hence $\varepsilon\kappa = \eta^2\bar{\kappa}$.