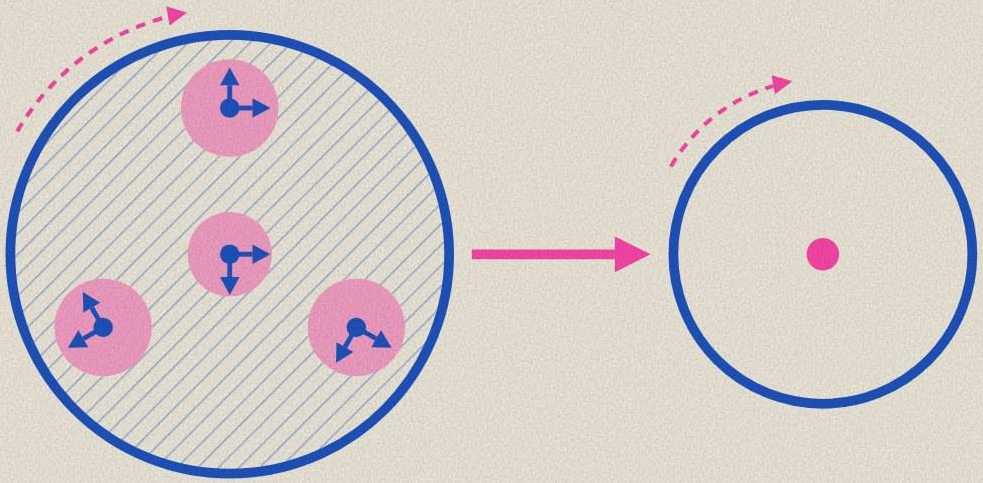
KYLE ORMSBY · REED COLLEGE · AMS WESTERN SECTIONAL · NOVEMBER 2026

Pontryagin–Thom, π_0 of the motivic sphere spectrum, and principal quadratic divisors

Segal's five-step computation of $\pi_0^G(\mathbb{S})$, carried out over a field.

arXiv:2610.29065

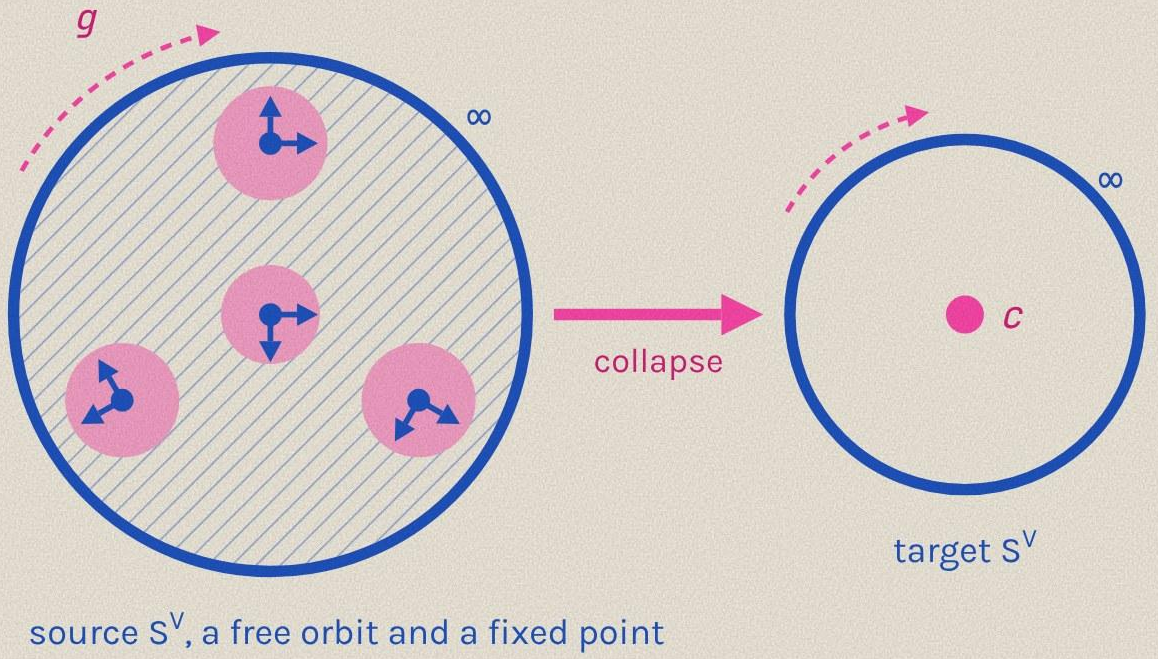


PONTRYAGIN · THOM · SEGAL

Pontryagin–Thom for framed G-manifolds

$f\colon S^V \to S^V$ equivariant, c a regular value: $f^{-1}(c)$ is a finite G-set, **framed** by df.

Conversely, a framed finite G-set gives a map: each framed disc onto S^V, everything else to ∞.



$$\pi_0^G(\mathbb{S}) = \operatorname*{colim}_V [S^V, S^V]^G \cong \{\text{framed finite } G\text{-sets}\}/\text{framed cobordism}$$

SEGAL (1971)

Five stops from $A(G)$ to $\pi_0^G(\mathbb{S})$

Theorem. PT: $A(G) \xrightarrow{\cong} \pi_0^G(\mathbb{S})$, a finite G-set to the collapse map of its orbits.

S1 collapse

Framed orbit ↦ stable map.

S2 isotopy

The class is well defined.

S3 ring

$\sqcup \mapsto +$, $\times \mapsto \wedge$.

S4 transversality

Every map is a collapse map: **onto**.

S5 marks

Fixed-point degrees $|S^H|$ are injective: **one-to-one**.

Filled stops: the two that change character over a field.

Same five stops, with polynomials instead of manifolds.

THE THEOREM · k INFINITE PERFECT, $\operatorname{char} k \neq 2$

Segal's theorem over a field

$$\mathrm{PT}\colon A^u(\Gamma_k) \longrightarrow \pi_0(\mathbb{S}_k)(k)$$

$(L, u) \mapsto$ a framed point with residue field L and framing determinant u,

is a **surjective ring homomorphism**,

with kernel the kernel of the **Dress map** $d(L,u) = \mathrm{tr}_{L/k}\langle u \rangle$,

which is generated by **principal quadratic divisors** of rational functions on $\mathbb{P}^1_k$.

$$\pi_0(\mathbb{S}_k)(k) \;\cong\; A^u(\Gamma_k) \,/\, (\text{principal quadratic divisors}) \;\cong\; \mathrm{GW}(k)$$

THE SOURCE AND THE MARKS

Finite étale schemes, decorated with a unit

$A^u(\Gamma_k)$ · unit-decorated Burnside ring

Generators (L, u): L/k finite separable, $u \in L^\times$.

$$(X, u) \otimes (Y, v) = (X \times_k Y,\ u \otimes v).$$

Dress's monomial Burnside ring $\Omega(\Gamma_k, (k^{\mathrm{sep}})^\times)$. No relation among units is imposed.

$d\colon A^u(\Gamma_k) \to \mathrm{GW}(k)$ · Dress map

$$d(L, u) = \mathrm{tr}_{L/k}\langle u\rangle\colon\ z \mapsto \mathrm{tr}_{L/k}(uz^2)$$

The trace form of the rank-one form $\langle u\rangle$: a ring map, onto, and far from injective.

Segal: $$A(G) \xrightarrow[\cong]{\text{PT}} \pi_0^G(\mathbb{S}) \xrightarrow[\text{one-to-one}]{\text{degrees}} \prod_{(H)} \mathbb{Z}$$

here: $$A^u(\Gamma_k) \xrightarrow[\text{onto}]{\text{PT}} \pi_0(\mathbb{S}_k)(k) \xrightarrow[\cong]{w} \mathrm{GW}(k)$$

Same shape; the isomorphism has moved one step to the right.

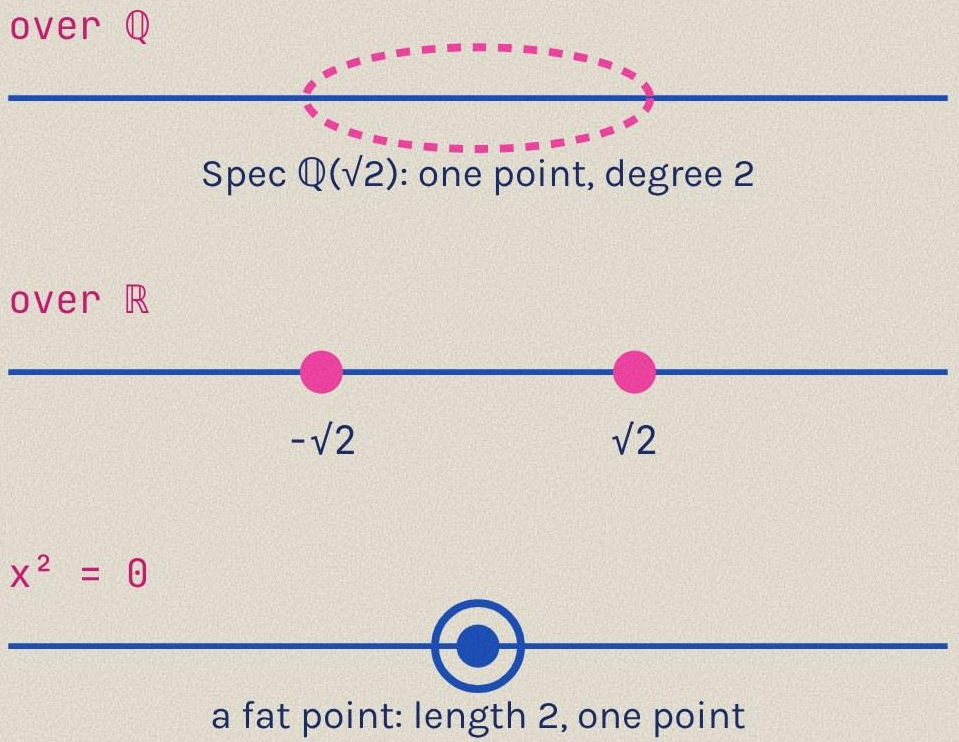
THE RULES OF THE GAME

Three things change

1. Homotopies are polynomial. Families over the whole t-line; no small perturbations.

2. Points have fields. A point with residue field L is an orbit of Γ_k.

3. Points can be fat. A framed zero-dimensional scheme may be non-reduced.



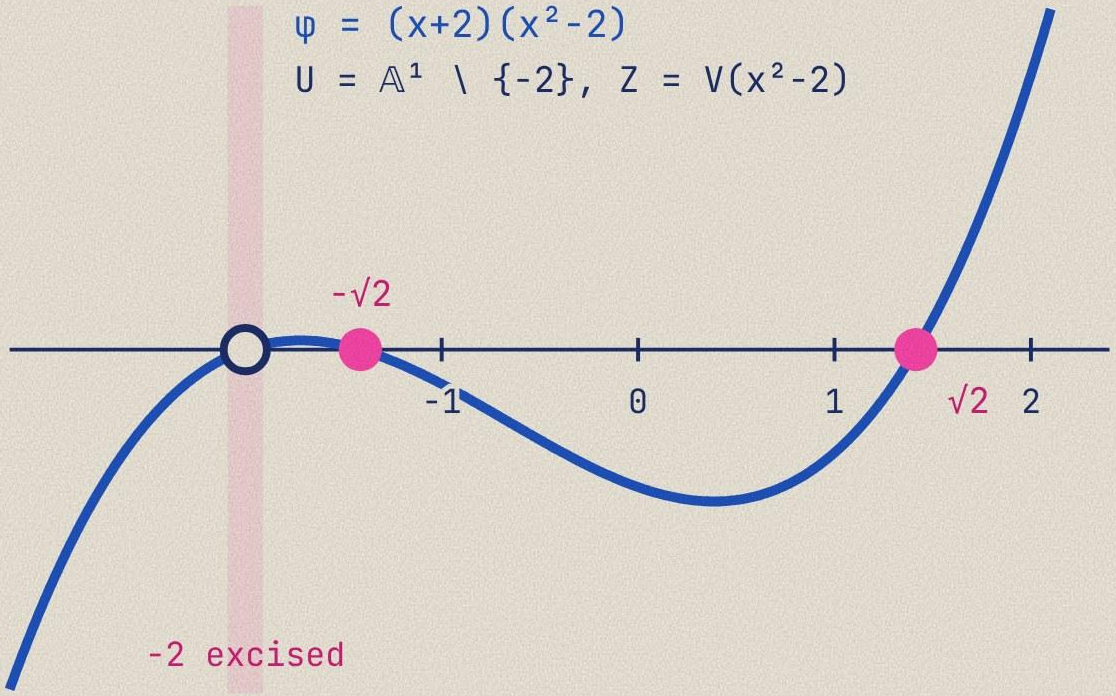
STOP S1 · A RUNNING EXAMPLE

A framed correspondence of level n

$$(Z,\ U,\ \varphi)$$

$Z \subset \mathbb{A}^n_k$ finite, U an étale neighborhood of Z, $\varphi\colon U \to \mathbb{A}^n$ with $V(\varphi) = Z$ exactly.

Here $\varphi = (x+2)(x^2-2)$ on $U = \mathbb{A}^1 \smallsetminus \{-2\}$, so $Z = \operatorname{Spec} \mathbb{Q}(\sqrt{2})$.



STOP S2 · WHERE THE SIGN GOES

The sign becomes a unit

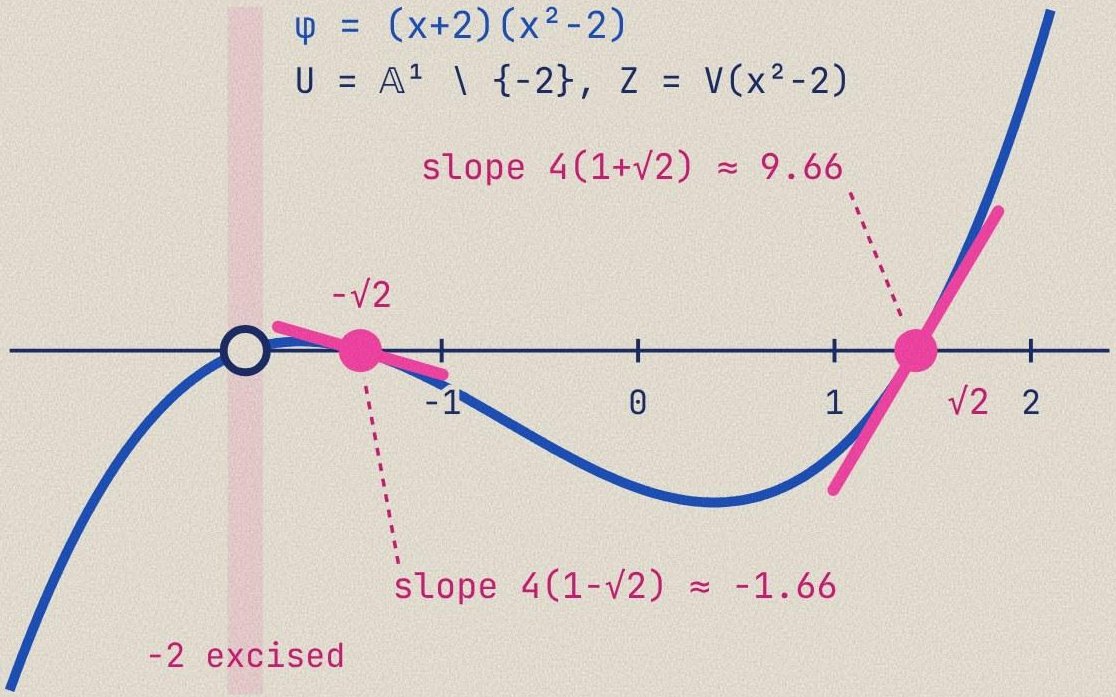
Topology: the framing at a point is a **sign**.

Here: the slope $\varphi'(\sqrt{2}) = 4(1+\sqrt{2})$, a **unit of the residue field**.

Only its value on Z matters (framing rigidity), and only its class mod squares will survive:

$$(\mathbb{Q}(\sqrt{2}),\ 1+\sqrt{2})$$

$e(L,u) \in \pi_0^{\mathrm{fr}}(k)$: the class of any framed point with support $\operatorname{Spec} L$ and framing determinant $\det d\varphi|_Z = u$. $\mathrm{PT}(L,u) := e(L,u)$, a ring map.



THE INPUT FROM MOTIVIC HOMOTOPY THEORY

Framed recognition

A level-n framed correspondence gives a map of motivic spheres

$$T^n \simeq \mathbb{P}^n/\mathbb{P}^{n-1} \xrightarrow{\text{collapse}} \mathbb{A}^n/(\mathbb{A}^n \smallsetminus Z) \xrightarrow[\simeq]{\text{excision}} U/(U \smallsetminus Z) \xrightarrow{\varphi} T^n$$

$T = \mathbb{A}^1/(\mathbb{A}^1 \smallsetminus 0)$; Z is finite, so it misses the hyperplane at infinity; $U \to \mathbb{A}^n$ is an isomorphism over Z.

hence an element of $\pi_0(\mathbb{S}_k)(k) = [\mathbb{S}_k, \mathbb{S}_k]$. Raising the level, $(Z, U, \varphi) \mapsto (Z \times 0,\, U \times \mathbb{A}^1,\, (\varphi, t))$, smashes this map with T.

INPUT · GARKUSHA–PANIN, ELMANTO–HOYOIS–KHAN–SOSNILO–YAKERSON

For k infinite perfect, $\pi_0^{\mathrm{fr}}(k) \xrightarrow{\cong} \pi_0(\mathbb{S}_k)(k)$, where $\pi_0^{\mathrm{fr}}(k)$ is the free abelian group on framed correspondences of all levels, modulo $Z' \sqcup Z'' = Z' + Z''$, level raising, and $\mathbb{A}^1$-homotopy.

An $\mathbb{A}^1$-homotopy is a framed correspondence over $\mathbb{A}^1_k$ whose support is finite over $\mathbb{A}^1$; it identifies its fibers at $t = 0$ and $t = 1$.

Every ingredient has a translation

finite group G	→	absolute Galois group Γ_k
orbit G/H	→	$\mathrm{Spec}\, L$, L/k finite separable
framing of a point (a sign)	→	unit $u \in L^\times$, the framing determinant
Burnside ring $A(G)$	→	unit-decorated Burnside ring $A^u(\Gamma_k)$
$\pi_0^G(\mathbb{S})$ = framed G-bordism	→	framed recognition (Garkusha–Panin, EHKSY)
fixed-point degree	→	Scheja–Storch form w, a Bézoutian
marks $\lvert S^H \rvert$	→	Dress map $d(L,u) = \mathrm{tr}_{L/k}\langle u \rangle$
equivariant transversality	→	Position Lemma, by re-topping
marks injective (Burnside)	→	every étale class splits into rational ones (Cazanave)

S1–S3 are done. The last four rows are the rest of the talk.

The fixed-point degree is a quadratic form

$\mathcal{O}(Z) = \mathcal{O}(U)/(\varphi)$ is a complete intersection. It carries a canonical functional $\eta\colon \mathcal{O}(Z) \to k$ with $\eta(\det d\varphi) = \dim_k \mathcal{O}(Z)$, built from the Bézoutian $\det(H_{ij})$, $\varphi_i(x) - \varphi_i(y) = \sum_j (x_j - y_j) H_{ij}$.

$$w(Z, U, \varphi) \;:=\; \big[(a,b) \mapsto \eta(ab)\big] \;\in\; \mathrm{GW}(k)$$

nondegenerate

a form on $\mathcal{O}(Z)$, of rank $\dim \mathcal{O}(Z)$

$\mathbb{A}^1$-invariant

Harder: forms over $k[t]$ are extended from k; so $w\colon \pi_0^{\mathrm{fr}}(k) \to \mathrm{GW}(k)$, a ring map

$w \circ \mathrm{PT} = d$

for Z étale, $\eta = \mathrm{tr}(\,\cdot\,/\det d\varphi)$, so $w(e(L,u)) = \mathrm{tr}_{L/k}\langle u \rangle$

$w(e(\mathbb{Q}(\sqrt{2}), 1+\sqrt{2}))$, two ways

As a Bézoutian. Write the point as A/B with $B(\theta) = A'(\theta)/u$:

$$A = x^2 - 2, \qquad B(\theta) = \frac{2\theta}{1+\theta} = 4 - 2\theta$$

$$\frac{A(x)B(y) - A(y)B(x)}{x - y} = 4(x+y) - 4 - 2xy$$

so in the basis $1, x$:

$$\begin{pmatrix} -4 & 4 \\ 4 & -2 \end{pmatrix} \cong \langle -4, 2 \rangle = \langle 2, -1 \rangle$$

As a trace form. With $z = x + y\sqrt{2}$:

$$\begin{aligned} \operatorname{tr}\left((1+\sqrt{2})z^2\right) &= 2x^2 + 8xy + 4y^2 \\ &= 2(x+2y)^2 - 4y^2 \end{aligned}$$

Same class $\langle 2, -1 \rangle$.

Over $\mathbb{R}$: signature 0, the two slopes have opposite signs. Over $\mathbb{Q}$: not hyperbolic, since 2 is not a square.

Fatness carries no information

POSITION LEMMA

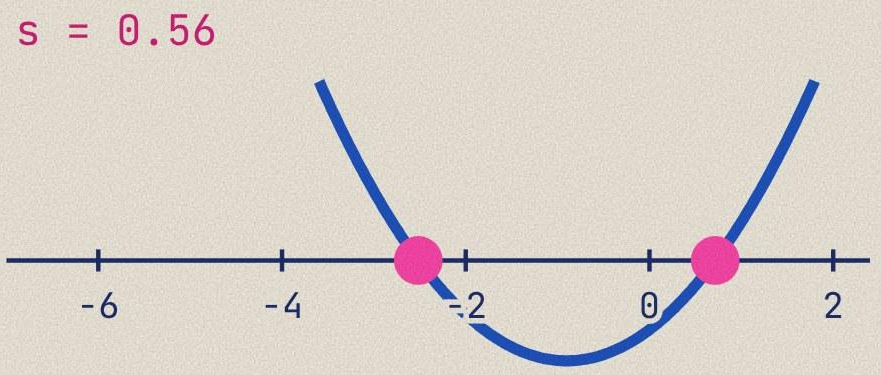
Every class in $\pi_0^{\mathrm{fr}}(k)$ is a $\mathbb{Z}$-combination of classes with finite **étale** support. So PT is onto.

One variable, $\varphi = x^2$ at the origin:

$x^2(1 - tx)$ – a zero escapes at $1/t$: **not** a homotopy.

$x^2 - t$ – finite flat over the t-line, étale at $t = 1$.

The difference: the **top-degree form** x^2 was held fixed.



$y = sx^2 + x - 1$: as $s \to 0^+$ one zero runs off to $-\infty$.

Re-topping

1 · polynomial framing

Reduce to φ polynomial at a single point, generating its ideal I.

2 · re-top

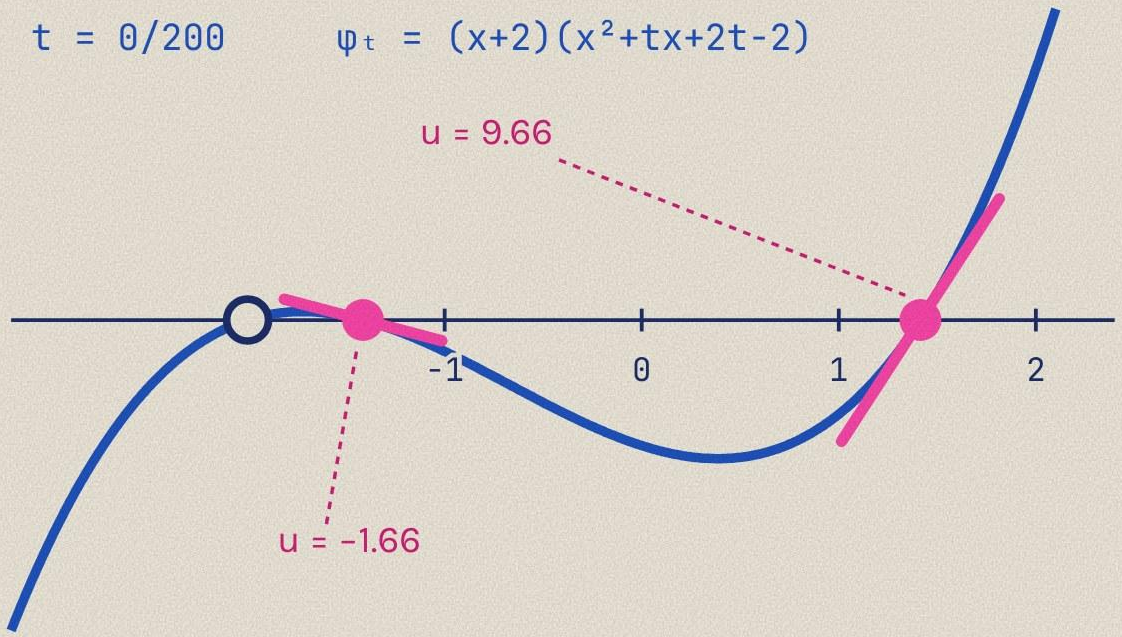
I/I^2 is free on φ (complete intersection), so by Nakayama $x_j^M \equiv g_j \bmod I^2$ with $\deg g_j < M$. Replace φ by ψ with top forms $x_1^M, \ldots, x_n^M$: same support, same class.

3 · deform

$\psi - t\gamma$ is finite flat over $\mathbb{A}^1$: the top forms do not depend on t, so nothing escapes. For generic γ the fiber at $t = 1$ is étale.

The framing is essential: an unframed fat point degenerates to a square-zero scheme and then to reduced points, which would collapse π_0 to the rank. Square-zero schemes are not complete intersections.

One irrational point splits into two rational ones



t

0 | 7/40 | 7/10 | 1 | animate

fiber over $\mathbb{Q}$: **one point** of degree 2
residue field $\mathbb{Q}(\sqrt{2})$
over $\mathbb{R}$: slopes 9.66 and -1.66
$\mathrm{tr}(u\cdot(a+b\theta)^2) = D\cdot[(a+2\cdot b)^2 - 2b^2]$
$D = t^2 - 8t + 8 = 8$
$D = (t-4)^2 - 2\cdot 2^2$, a norm from $\mathbb{Q}(\sqrt{2})$,
so the class is $\langle 1,-2\rangle \cong \langle 2,-1\rangle$ for every t

$$\varphi_t = (x+2)\left(x^2+tx+2t-2\right)$$

No zero reaches -2, none escapes to infinity: a legal $\mathbb{A}^1$-homotopy. The residue field jumps; the form does not.

$$e(\mathbb{Q}(\sqrt{2}), 1+\sqrt{2}) = e(\mathbb{Q}, 2) + e(\mathbb{Q}, -1)$$

Pointed rational functions, after Cazanave

A/B with A monic of degree n and $\deg B < n$ is a framed correspondence $\big(V(A),\ \mathbb{A}^1 \smallsetminus V(B),\ A/B\big)$, and a naive homotopy of such functions is an $\mathbb{A}^1$-homotopy.

INPUT · CAZANAVE (2012)

Naive homotopy classes of pointed rational functions of degree n are classified by the pair

$$\big(\ \mathrm{Béz}(A,B),\ \ \mathrm{res}(A,B)\ \big)\ \in\ \mathrm{GW}(k)\times k^{\times}.$$

The framed point $e(L,u)$ is A/B with A the minimal polynomial and $B(\theta)=A'(\theta)/u$; its Bézout form is $\mathrm{tr}_{L/k}\langle u\rangle$.

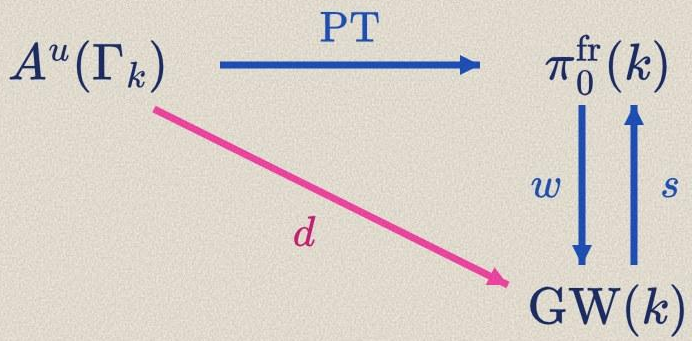
Every étale point splits into rational ones

MARKS THEOREM

$e(L,u) = \sum_i e(k,v_i)$ whenever $\mathrm{tr}_{L/k}\langle u\rangle \cong \langle v_1,\dots,v_n\rangle$.

So $s\langle v\rangle := e(k,v)$ is a well-defined section of w, and $\mathrm{PT} = s \circ d$.

Proof: match the Bézout form with a split function (distinct rational zeros, units v_i), match the resultant by rescaling one v_i by a square, apply Cazanave. Witt's relations among the $\langle v\rangle$ are realized by explicit degree-two pencils.

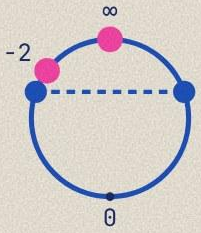


$w \circ \mathrm{PT} = d \cdot w \circ s = \mathrm{id} \cdot \mathrm{PT} = s \circ d$

$w \circ s = \mathrm{id};\ s \circ w = \mathrm{id}$ on étale classes, hence everywhere by S4. So $\pi_0^{\mathrm{fr}}(k) \cong \mathrm{GW}(k)$ and $\ker \mathrm{PT} = \ker d$.

What does the Dress map forget?

Classical. A rational function on $\mathbb{P}^1$ has as many zeros as poles, with multiplicity and degree.



$f = (x+2)/(x^2-2)$ on $\mathbb{P}^1_{\mathbb{Q}}$:

- zeros -2 and ∞, degree $1+1$
- pole $x^2 = 2$: *one* point of degree 2, order 1

Quadratic (Scharlau 1972). At each zero or pole p of *odd order* take its slope $u_p \in \kappa(p)^\times$. Then

$$\sum_{\substack{p \neq \infty \\ v_p(f)\ \mathrm{odd}}} \mathrm{tr}_{\kappa(p)/k}\langle u_p \rangle \;=\; [v_\infty(f)\ \mathrm{odd}]\,\langle u_\infty \rangle + r\,h$$

$[\,\cdot\,]$ the Iverson bracket; r fixed by ranks; $h = \langle 1, -1 \rangle$. At a simple zero $u_p = f'(p)$, at a simple pole $(1/f)'(p)$; at ∞ use the coordinate $1/x$.

$\ker d$ is cut out by rational functions

THEOREM

$\ker(d)$ is generated by the $R(f)$, $f \in k(x)^\times$: the odd-order zeros and poles of f with their slopes, minus ∞ if of odd order, minus r copies of $h_A = (k,1) + (k,-1)$.

Example, $f = (x+2)/(x^2-2)$. Apply d to $R(f)$:

$$\underbrace{\langle 2\rangle + \langle -1\rangle}_{\text{pole } x^2=2} + \underbrace{\langle 2\rangle}_{\text{zero } -2} - \underbrace{\langle 1\rangle}_{\text{zero } \infty} - \underbrace{\big(\langle 1\rangle + \langle -1\rangle\big)}_{h} \;=\; 2\langle 2\rangle - 2\langle 1\rangle \;=\; 0$$

because $2x^2 + 2y^2 = (x+y)^2 + (x-y)^2$.

$$\pi_0(\mathbb{S}_k)(k) \cong A^u(\Gamma_k) \,/\, \text{principal quadratic divisors}$$

OUTLOOK · WITH A GROUP ACTING (IN PREPARATION)

The same five stops for $\pi_0^G(\mathbb{S}_k)$

Source $A^u(G;k)$: finite étale G-schemes with a G-invariant unit. Marks: geometric fixed points, then the Scheja–Storch degree.

THEOREM (90%) · $|G| \in k^\times$

$\mathrm{PT}\colon A^u(G;k) \to \pi_0^G(\mathbb{S}_k)$ is onto (via Gepner–Heller's tom Dieck splitting). For G split abelian, the universal Kummer torsor gives

$$\pi_0^G(\mathbb{S}_k) \;\cong\; \bigoplus_{H \le G} \; \bigoplus_{S} \; K^{MW}_{|S|}(k)/(n_{i,\varepsilon} : i \in S),$$

and $\ker \mathrm{PT}$ is the kernel of the Milnor–Witt marks. Here $G/H = \prod_i \mu_{n_i}$, S runs over sets of factors, and $n_\varepsilon = \sum_{j=0}^{n-1} \langle (-1)^j \rangle \in \mathrm{GW}(k)$ is the Milnor–Witt integer, so that $[a^n] = n_\varepsilon [a]$.

$G = C_2$

$\mathrm{GW} \oplus \mathrm{GW} \oplus I$; in the quadratic marks (q_+, q_-, r) the product is componentwise.

$G = \mu_\ell$, ℓ odd

$\{(a,r) \in \mathrm{GW}^2 : \ \mathrm{rk}\, a \equiv \mathrm{rk}\, r \bmod \ell\} \oplus k^\times/\ell$: tom Dieck's congruence over GW, plus a Kummer class invisible to quadratic marks.

relations

Pencils carrying torsors; checked by computer against the kernel of the marks over $\mathbb{Q}_p$.

THANK YOU

Count the points.
Keep the units.
Divide by principal quadratic divisors.

$$\pi_0(\mathbb{S}_k)(k) \;\cong\; A^u(\Gamma_k) \,/\, \ker(d) \;\cong\; \mathrm{GW}(k)$$

arXiv:2610.29065

Partially supported by NSF award DMS-2604878.

REFERENCES · AI USE

Bibliography

THIS TALK

K. Ormsby, *Pontryagin–Thom, π_0 of the motivic sphere spectrum, and principal quadratic divisors*, arXiv:2610.29065.

TOPOLOGY

L. S. Pontryagin, *Smooth manifolds and their applications in homotopy theory*, Amer. Math. Soc. Transl. (2) 11 (1959), 1–114.

R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. 28 (1954), 17–86.

G. Segal, *Equivariant stable homotopy theory*, Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, 1971, 59–63.

THE MOTIVIC SPHERE

F. Morel, *On the motivic π_0 of the sphere spectrum*, NATO Sci. Ser. II 131, Kluwer, 2004, 219–260.

F. Morel, *$\mathbb{A}^1$-algebraic topology over a field*, Lecture Notes in Math. 2052, Springer, 2012.

QUADRATIC FORMS AND DEGREES

A. W. M. Dress, *The ring of monomial representations I*, J. Algebra 18 (1971), 137–157.

W. Scharlau, *Quadratic reciprocity laws*, J. Number Theory 4 (1972), 78–97.

T. Y. Lam, *Serre's problem on projective modules*, Springer, 2006 (Ch. VII, Thm. 3.13: Harder's theorem).

G. Scheja and U. Storch, *Über Spurfunktionen bei vollständigen Durchschnitten*, J. Reine Angew. Math. 278/279 (1975), 174–190.

D. Eisenbud and H. I. Levine, *An algebraic formula for the degree of a C^∞ map germ*, Ann. of Math. (2) 106 (1977), 19–44.

G. N. Khimshiashvili, *On the local degree of a smooth mapping*, Soobshch. Akad. Nauk Gruzin. SSR 85 (1977), 309–312.

C. Cazanave, *Algebraic homotopy classes of rational functions*, Ann. Sci. Éc. Norm. Supér. 45 (2012), 511–534.

FRAMED CORRESPONDENCES

V. Voevodsky, *Notes on framed correspondences*, unpublished, 2001.

G. Garkusha and I. Panin, *Framed motives of algebraic varieties (after V. Voevodsky)*, J. Amer. Math. Soc. 34 (2021), 261–313.

E. Elmanto, M. Hoyois, A. A. Khan, V. Sosnilo, and M. Yakerson, *Motivic infinite loop spaces*, Camb. J. Math. 9 (2021), 431–549.

T. Bachmann, E. Elmanto, M. Hoyois, A. A. Khan, V. Sosnilo, and M. Yakerson, *On the infinite loop spaces of algebraic cobordism and the motivic sphere*, Épijournal Géom. Algébrique 5 (2021), Art. 9.

A. Neshitov, *Framed correspondences and the Milnor–Witt K-theory*, J. Inst. Math. Jussieu 17 (2018), 823–852.

A. Druzhinin and J. I. Kylling, *Framed correspondences and the zeroth stable motivic homotopy group in odd characteristic*, arXiv:1809.03238.

M. Hoyois, J. Jelisiejew, D. Nardin, B. Totaro, and M. Yakerson, *The Hilbert scheme of infinite affine space and algebraic K-theory*, J. Eur. Math. Soc. 27 (2025), 2037–2071.

AI use. AI assistants (Claude) were used to help draft these slides and to develop parts of the mathematics in this project, in particular the re-topping proof of the Position Lemma; see the disclosure in the paper. The author takes full responsibility for the mathematical content.

BACKUP A1 · STOPS S1–S3 IN DETAIL

The map, and why it is well defined

S1 (collapse). For $L = k(\theta)$ with minimal polynomial A and $u \in L^\times$, let $e(L,u)$ be the class of $\big(V(A),\ \mathbb{A}^1 \smallsetminus V(B),\ A/B\big)$ with $B(\theta) = A'(\theta)/u$, whose framing determinant at θ is $A'(\theta)/B(\theta) = u$. Set $\mathrm{PT}(L,u) = e(L,u)$.

S2 (isotopy). Framing rigidity: two framings of the same étale Z with the same Jacobian determinant on Z give the same class. Shears, SL_n, and simultaneous changes of coordinates act trivially, so the class depends only on $\det d\varphi|_Z \in \mathcal{O}(Z)^\times$.

S3 (ring). $\sqcup \mapsto +$, and the external product of (L,u) and (L',u') has support $\mathrm{Spec}(L \otimes_k L')$ with unit $u \otimes u'$. PT is a ring map.

What survives. The homotopy $(\varphi_1(1+t\nu), \varphi_2, \ldots), \nu$ vanishing on Z, is free, so only the value of $\det d\varphi$ on Z matters: a unit in $\mathcal{O}(Z)^\times$. Nothing here identifies u with uc^2; that happens at S5, and it takes a second point.

The substitute for "$GL_n^+(\mathbb{R})$ is connected": $SL_n = E_n$ over commutative semilocal rings (Bass), and elementary matrices are $\mathbb{A}^1$-homotopic to the identity.

Residues, the Euler unit, and $R(f)$

Points. A closed point $p \in \mathbb{A}^1_k$ has monic irreducible $\pi_p \in k[x]$, residue field $\kappa(p) = k[x]/(\pi_p)$, and root θ_p. At ∞ take $\pi_\infty = 1/x$ and $\kappa(\infty) = k$.

Euler unit. $\varepsilon_p := \pi_p'(\theta_p) \in \kappa(p)^\times$, the slope of the minimal polynomial at its root; $z \mapsto \mathrm{tr}(z/\varepsilon_p)$ is the functional dual to $\theta_p^{\deg \pi_p - 1}$ (Euler).

Leading unit. Write $f = \pi_p^{v_p(f)} g_p$ with g_p a unit at p, and set $\ell_p(f) := g_p(p) \in \kappa(p)^\times$.

Second residue. $\partial_p \colon \mathrm{W}(k(x)) \to \mathrm{W}(\kappa(p))$, $\partial_p\langle \pi_p^n g\rangle = \langle g(p)\rangle$ for n odd, 0 for n even.

Scharlau's reciprocity law (1972, Thm. 4.1). For every $q \in \mathrm{W}(k(x))$,

$$\sum_{p \in (\mathbb{A}^1_k)^{(1)}} \mathrm{tr}_{\kappa(p)/k}\left(\langle \varepsilon_p\rangle \otimes \partial_p q\right) \;=\; \partial_\infty q \quad \text{in } \mathrm{W}(k).$$

The Euler twist is essential: for $q = \langle x^2 - 2\rangle$ it turns the trace form $\langle 1, 2\rangle \neq 0$ into $\mathrm{tr}\langle\sqrt{2}\rangle$, which is hyperbolic.

Quadratic divisor. For $f \in k(x)^\times$,

$$\rho(f) := \sum_{\substack{p \neq \infty \\ v_p(f) \text{ odd}}} \big(\kappa(p),\, \varepsilon_p\, \ell_p(f)\big) - \big[v_\infty(f) \text{ odd}\big]\,\big(k,\, \ell_\infty(f)\big) \;\in A^u(\Gamma_k).$$

At a simple zero $\varepsilon_p \ell_p(f) = f'(\theta_p)$; at a simple pole it is $(1/f)'(\theta_p)$ up to squares. These are the slopes of the kernel slide.

Rank correction. $\deg \mathrm{div}(f) = 0$ makes $\mathrm{rank}\,\rho(f) = 2r(f)$ even. The *principal quadratic divisor* of f is $R(f) := \rho(f) - r(f)\, h_A$, with $h_A = (k, 1) + (k, -1)$.

> **THEOREM**
> For k perfect of characteristic $\neq 2$ (infinite not needed here), $\ker(d)$ is the subgroup generated by the $R(f)$.

Proof outline. $R(f) \in \ker d$: apply Scharlau's law to $q = \langle f\rangle$; then $d(\rho(f)) = 0$ in $\mathrm{W}(k)$, so it is $r(f)h$ in $\mathrm{GW}(k)$. $\ker d \subseteq \langle R(f)\rangle$: every class is congruent mod R to one on rational points, and Witt's relations among the $\langle u\rangle$ are themselves of the form $R(f)$.

BACKUP A3 · WHY A KERNEL AT ALL

The Burnside source has to be cut down

Segal's map $A(G) \to \pi_0^G(\mathbb{S})$ is an isomorphism. Ours is onto, with a kernel. Why?

Segal uses only *canonical* framings. A point with another framing is not a new generator; it is expressed through the others. Example: C_2 acting on $\mathbb{R}$ by $x \mapsto -x$. The fixed point with the opposite framing is $[\mathrm{pt}] - [C_2/e]$; both have marks $(-1, 1)$.

$A^u(\Gamma_k)$ keeps every unit as a separate generator, so it must be cut down. The kernel records how a point with an arbitrary unit is expressed through others. Those expressions come from families that change the type of the points, as in the demo, and they are exactly the principal quadratic divisors.

The reading of the theorem. Quadratic forms over k are what you get from finite étale k-schemes with units, if you let points split and merge along rational functions on the line.

Over $\mathbb{R}$ there are two kinds of relation:

$$(\mathbb{R}, u) \sim (\mathbb{R}, uc^2) \qquad \text{squares},$$
$$(\mathbb{C}, u) \sim (\mathbb{R}, 1) + (\mathbb{R}, -1) \qquad \text{a } \pm \text{ pair}.$$

That leaves rank and signature: a ± pair of real preimages that is born or dies is a complex-conjugate pair landing on the real line and splitting, or the reverse.

Over $\mathbb{Q}$: rank, signature, discriminant and Hasse invariants at every prime, and every relation comes from a rational function on $\mathbb{P}^1_{\mathbb{Q}}$.