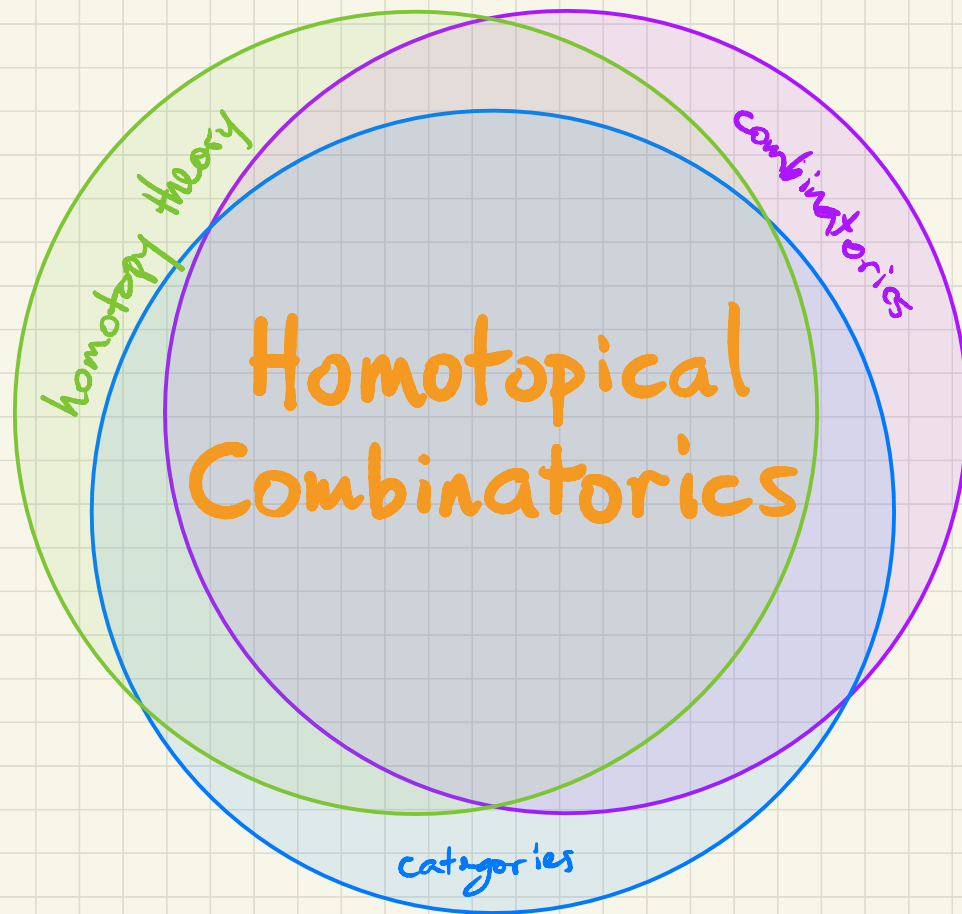




Kyle Ormsby
25.8.22

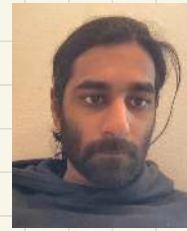
Joint with



Scott Balchin
MPIM



Evan Franchere
UKy [Read]



Usman Hafeez
[Read]



Ethan MacBrough
Read



Peter Marcus
Tulane [Read]



Angélica Osorno
Read



Weihang Qin
[Read]



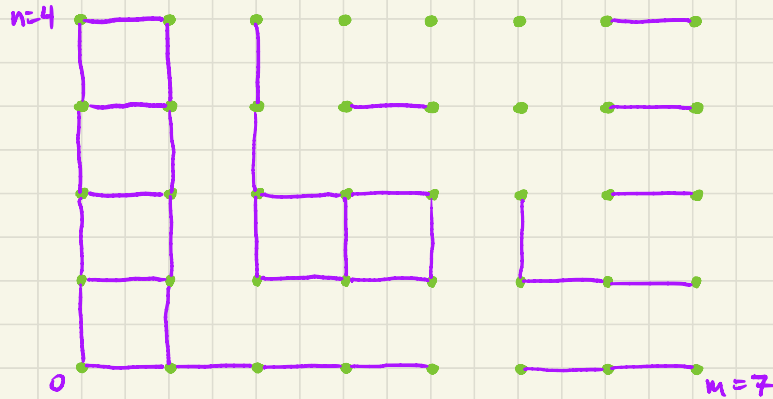
Constanze Roitzheim
Kent



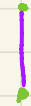
Riley Waugh
[Read]

Warmup

The Matchstick Game



Rules



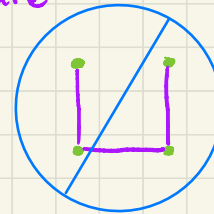
$\Rightarrow$ all verticals to the left



$\Rightarrow$ all horizontals below

3 sides of a
unit square

$\Rightarrow$ full unit
square



(and rotations)

Q1 How many legal matchstick configurations on $[n] \times [n]$?

Q2 What else do these count?

Quillen model structures

A model structure on a category $\mathcal{C}$ consists of three classes of morphisms:

W = weak equivalences

F = fibrations

C = cofibrations

$$\left(\begin{array}{l} AF = W \cap F = \text{acyclic fibrations} \\ AC = W \cap C = \text{acyclic cofibrations} \end{array} \right)$$

satisfying five axioms:

MC1) $\mathcal{C}$ is complete & cocomplete

MC2) W satisfies $2 \Rightarrow 3$: two of $f, g, fg \in W \Rightarrow$ all in W

MC3) W, F, C are closed under retracts (in the arrow category)

MC4) $C \boxtimes F$ (lifting condition defined later)

MC5) $AF \circ C = \text{Mor}(\mathcal{C}) = F \circ AC$ (factorization)

Theorems

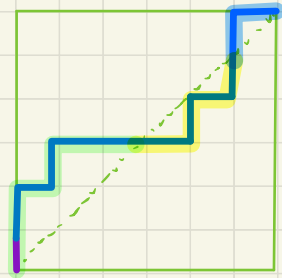
Goal Given a (co)complete category $\mathcal{C}$, classify and enumerate all model structures on $\mathcal{C}$.

Thm (Folklore/Goodwillie/Barthel-Antolin Camarino) There are precisely nine model structures on $\mathbf{Set}$.

Thm (Balchin-O-Osorno-Roitzeim) There are precisely

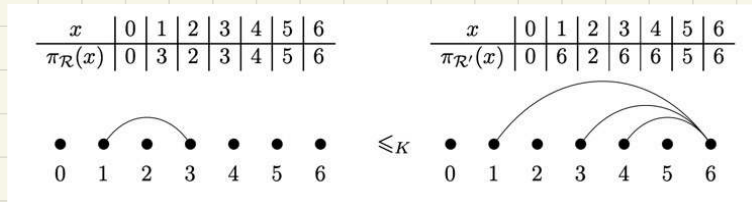
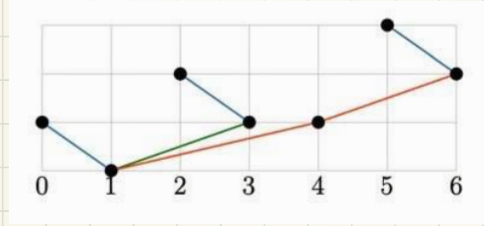
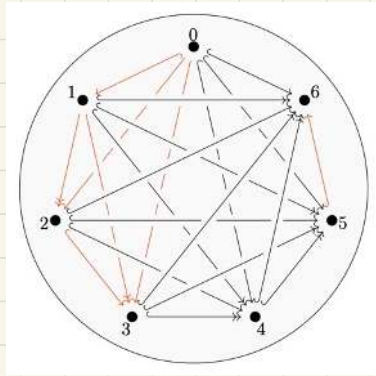
$\binom{2n+1}{n}$ model structures on $[n] := \{0 \rightarrow 1 \rightarrow 2 \rightarrow \dots \rightarrow n\}$.

For $0 \leq k \leq n$, precisely $\frac{2(k+1)}{n+k+2} \binom{2n+1}{n-k}$ of these have homotopy category $\cong [k]$.

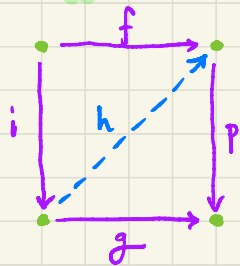


Theorems

Thm (Balchin-O-MacBrough) Moreover, model structures on $[n]$ are in bijection with (a) model triangulations, (b) model tricolored trees, and (c) model intervals in the Kreweras lattice of noncrossing partitions.



Lifting + Weak Factorization Systems



Say i has the left lifting property wrt p
 (and p has the right lifting property wrt i)
 and write $i \Box p$ when for $\forall f, g$ s.t. $\begin{array}{ccc} & f & \\ i \downarrow & \rightarrow & \downarrow p \\ & g & \end{array}$ commutes,
 $\exists h$ such that $f = hi$, $g = ph$.

For $M \subseteq \text{Mor } C$, write $\Box M := \{i \mid i \Box p \ \forall p \in M\}$

$M^\Box := \{p \mid i \Box p \ \forall i \in M\}$.

A weak factorization system on C consists of $(L, R) \subseteq \text{Mor } C \times \text{Mor } C$
 such that

WF1) $\text{Mor } C = R \circ L$ i.e. $\begin{array}{ccc} & f & \\ i \swarrow & \rightarrow & \searrow p \\ & x & \end{array}$ with $i \in L, p \in R$,

WF2) $L \Box R$ i.e. $L \in \Box R$ and $R \in L^\Box$, and

WF3) L, R closed under retracts.

Premodel Structures

Defn (Barton) A premodel structure on a (co)complete category $\mathcal{C}$ is a pair of weak factorization systems

$$\begin{array}{c} (\mathcal{C}, AF) \\ \Downarrow \quad \Downarrow \\ (AC, F) \end{array} \quad \text{with } \mathcal{C} = \square AF, \quad AC = \square F.$$

Call $AF =:$ anodyne fibrations,
 $AC =:$ anodyne cofibrations.

Every model structure (W, F, C) induces a premodel structure

$$\begin{array}{c} (\mathcal{C}, W \cap F) \\ \Downarrow \quad \Downarrow \\ (W \cap C, F), \end{array} \quad \text{In this scenario, } W = AF \circ AC.$$

Joyal-Tierney presentation of model structures

Thm (Joyal-Tierney) For a (co)complete category C ,

$$\left\{ \begin{array}{l} (C, AF) \\ \text{IU} \quad \text{II} \\ (Ac, F) \end{array} \right. \begin{array}{l} \text{premodel structure} \\ \text{on } C \end{array} \mid \begin{array}{l} AF \circ AC \text{ satisfies} \\ 2 \Rightarrow 3 \end{array} \right\} \xrightarrow{\cong} \left\{ (W, F, C) \text{ model} \right. \\ \left. \text{structure on } C \right\}$$

Upshot Premodel structures on C
are intervals in the poset $WFS(C)$

(ordered by $(L, R) \leq (L', R') \Leftrightarrow R \subseteq R' \\ [\Leftrightarrow L' \subseteq L]$).

These are model structures iff
 $R \circ L'$ has $2 \Rightarrow 3$.

$$\begin{array}{l} (C, AF) \\ \text{IU} \quad \text{II} \\ (Ac, F) \end{array} \mapsto (AF \circ AC, F, C)$$

$$\begin{array}{l} (C, W \cap F) \\ \text{IU} \quad \text{II} \\ (W \cap C, F) \end{array} \longleftarrow (W, F, C).$$

Lattices

Producing all WFS's on a general C is hard!

A complete lattice is a poset admitting all

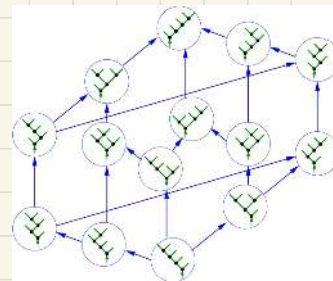
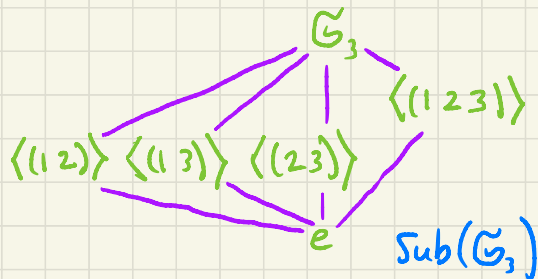
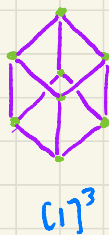
meets ($\wedge$ = infimum = limit)

& joins ($\vee$ = supremum = colimit).

E.g. Chain $[n] = \{0 < 1 < \dots < n\}$, Boolean lattice $[1]^n$, subgroup lattice $\text{Sub}(G)$, divisibility lattice, Tamari lattice, Kreweras lattice, ...

(rooted planar binary trees under rotation)

(noncrossing partitions under refinement)



Tamari

[Bernardi-Bonichon]

Lattice Categories

Henceforth, lattice = finite lattice = category induced thereby:

$$(P, \leq) \rightsquigarrow \text{Ob } P := P$$

$$P(x, y) = \begin{cases} \{x \xrightarrow{\exists!} y\} & \text{if } x \leq y \\ \emptyset & \text{otherwise.} \end{cases}$$

The category P is (co)complete
iff P is a complete lattice.

Refined Goal Enumerate and determine the structure of

$$\text{WFS}(P), \text{Pre}(P) \cong \text{Int}(\text{WFS}(P)), Q(P).$$

weak factorization
systems on P

premodel
structures on P

intervals

Quillen model structures
on P : subset of $\text{Pre}(P)$
with $W = \text{AF} \circ \text{AC}$ satisfying
 $2 \Rightarrow 3$.

$$\text{Int}(L) = \{(x, y) \in L^2 \mid x \leq y\} \text{ with}$$

$$(x, y) \leq (x', y') \text{ :iff } x \leq x' \text{ and } y \leq y'.$$

If L is a lattice, so is $\text{Int}(L)$.

Transfer systems

Blumberg-Hill:
equivariant DAG

To determine $\text{WFS}(P)$ we get an assist from ... N_∞ operads?!

Thm (Rubin et al) $H_0(G\text{-}N_\infty \text{ operads}) \cong \underbrace{\text{Tr}(\text{Sub}(G))}_{\text{transfer systems}}.$

A transfer system on a lattice $(P, \leq)$ is a transitive relation R on P refining $\leq$ (so $aRb \Rightarrow a \leq b$) that is closed under restriction:

$$\begin{array}{ccc} x & & x \vee y \xrightarrow{\cong} x \\ \downarrow R & \Rightarrow & \downarrow R \\ y \xrightarrow{\cong} z & & y \xrightarrow{\cong} z. \end{array}$$

(If $P = \text{Sub}(G)$, also require R to be closed under conjugation.)

$$\text{Tr} \cong \text{WFS}$$

Thm (Franchere-O-Osorno-Qin-Waugh) For P a lattice, we have a lattice isomorphism $\text{Tr}(P) \xrightarrow{\cong} \text{WFS}(P)$.

$$R \longmapsto (\emptyset R, R)$$

$\left. \begin{array}{c} \text{subrelations} \\ \text{of } \leq \end{array} \right\} \circ \circ \circ \left\{ \begin{array}{c} \text{subset of Mor } P \end{array} \right.$

In case $P = [n]$, we get some immediate progress:

Thm (Balchin-Barnes-Reitzheim) $\text{Tr}[n] \cong \mathcal{A}_{n+1}$, the Tamari lattice of planar full binary trees with $n+2$ leaves. As such,

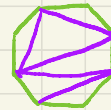
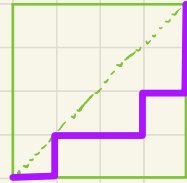
$$|\text{Tr}[n]| = \text{Cat}(n+1) = \frac{1}{n+2} \binom{2n+2}{n+1}.$$

Catalan #s

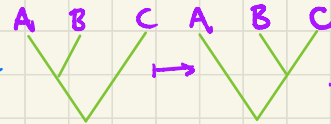
The Catalan numbers $\text{Cat}(n)$, $n \geq 0$, are the sequence 1, 1, 2, 5, 14, 42, 132, ...
 satisfying $\text{Cat}(n+1) = \sum_{i=0}^n \text{Cat}(i) \text{Cat}(n-i)$.

They enumerate

- planar full binary trees with $n+1$ leaves
- Dyck paths from $(0,0)$ to (n,n)
- noncrossing partitions of an n -element set
- triangulations of a convex $(n+2)$ -gon by chords
- much more!



The Tamari order on $\mathcal{A}_n$ is generated by tree rotation



The Kreweras order on $\underbrace{\text{NC}_n}_{\text{noncrossing partitions}}$ is given by refinement.

Model Structures on $[n]$

Amalgamating results, $\text{Pre } [n] \cong \text{Int}(\text{Tr } [n]) \cong \text{Int } \mathcal{A}_{n+1}$.

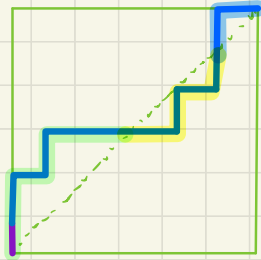
Thm (Chapoton) $|\text{Int } \mathcal{A}_n| = \frac{2}{n(n+1)} \binom{4n+1}{n-1}$.

Cor (BOUR) $|\text{Pre } [n]| = \frac{2}{(n+1)(n+2)} \binom{4n+5}{n}$.

Thm (BOUR) $|Q([n])| = \binom{2n+1}{n}$, and for $0 \leq k \leq n$, precisely $\frac{2(k+1)}{n+k+2} \binom{2n+1}{n-k}$ of these have homotopy category $\cong [k]$.

Pf Idea Specify k s.t. $[k] \cong \text{Ho}([n])$, then specify $k+1$ weak equiv classes, then count choices of contractible model structures:

$$|Q([n])| = \sum_{k=0}^n \underbrace{\sum_{\substack{i_0 + \dots + i_k \\ = n+1}} \prod_{j=0}^{k+1} \text{Cat}(i_j)}_{\text{Shapiro (1976)}} = \binom{2n+1}{n} \text{ by lattice paths}$$



CC Premodel Structures



Fill the gap between premodel & model structures.

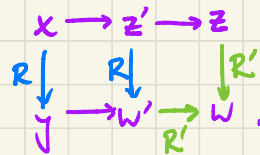
Call a premodel structure (C, AF) on C composition closed when
 $(\overset{u}{AC}, \overset{n}{F})$

$W := AF \circ AC$ is closed under composition. (Need $2 \Rightarrow 3$ for model str.)

For $C = P$ a lattice, write $R \leq R'$ for a premodel str / interval of transfer systems.

Thm (Balchin-MacBrough-0) For P a complete lattice, $\exists$ partial order $\leq$ on $WFS(P)$ refining $\leq$ and such that $R \leq R'$ if and only if $R \circ R'$ is closed under composition. Moreover, $R \leq R'$ iff

every square $x \rightarrow z$ has a splitting

$$\begin{array}{ccc} x & \rightarrow & z \\ R \downarrow & & \downarrow R' \\ y & \rightarrow & w \end{array}$$


CC Premodel Structures

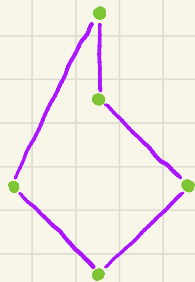
Thm (ct'd) If $\mathcal{P}$ is finite, then $(WFS(\mathcal{P}), \preceq)$ is a finite lattice. Thus

$CC(\mathcal{P})$ = $\text{Int}(WFS(\mathcal{P}), \preceq)$ is a finite lattice.

composition closed premodel structures

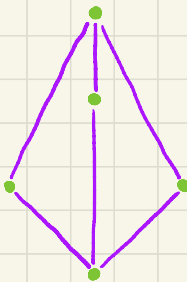
Note There is also an ordering $\sqsubseteq$ on $WFS(\mathcal{P})$ such that $\mathcal{R} \sqsubseteq \mathcal{R}'$ iff the pair forms a model structure, but $(WFS(\mathcal{P}), \sqsubseteq)$ is not a lattice.

E.g.



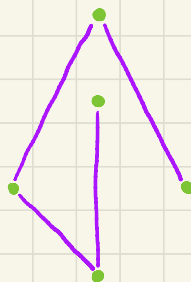
$(WFS([2]), \leq)$

$|\text{Pre}([2])| = 13$



$(WFS([2]), \preceq)$

$|CC([2])| = 12$



$(WFS([2]), \sqsubseteq)$

$|Q([2])| = 10$

Kreweras Intervals

For $R \in \text{Tr}[n]$, define $\pi_R: [n] \rightarrow [n]$
 $i \mapsto \max\{j \mid i R j\}.$

$[n] = B_1 \sqcup \dots \sqcup B_k$ is NC
 when $a, b \in B_i$, $x, y \in B_j$, and
 $a < x < b < y \Rightarrow i=j$.



Prop (FOOQW) Write NC_{n+1} for the set of noncrossing partitions of $\{0, 1, \dots, n\}$. Then $\text{Tr}[n] \rightarrow \text{NC}_{n+1}$

$$R \mapsto \{\pi_R^{-1}\{i\} \mid 0 \leq i \leq n, \pi_R^{-1}\{i\} \neq \emptyset\}$$

is a bijection.

Thm (BMO) Pulling back the Kreweras (refinement) order on NC_{n+1} to $\text{Tr}[n]$ gives the ordering $\leq$.

Cor $\text{CC}([n]) \cong \text{Int}(\text{NC}_{n+1})$ and thus

$$|\text{CC}([n])| = \frac{1}{2n+3} \binom{3n+3}{n+1}.$$

Kreweras: $= |\text{Int}(\text{NC}_{n+1})|$

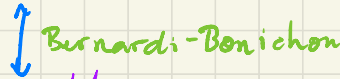
Kreweras Intervals

The following structures are equinumerous:

- Composition closed premodel structures on $[n]$



- Kreweras intervals in NC_{n+1}

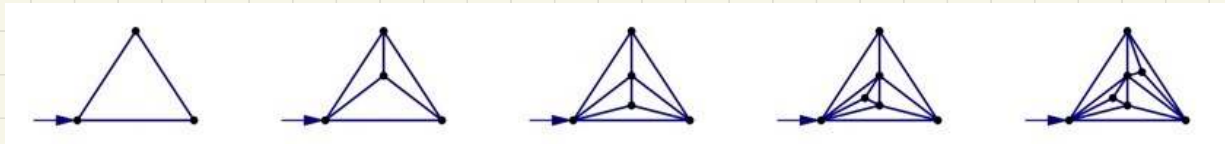


- (admissably, ordered) ternary trees on $n+1$ nodes

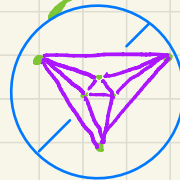


- stacked triangulations

} identify model str's among these

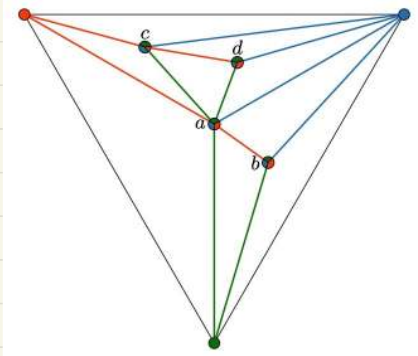


Stacked Δ 's formed by recursively inserting degree 3 vxs

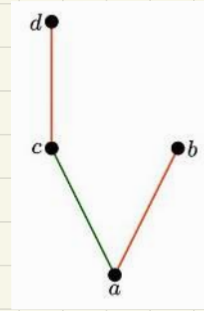


not stacked

$\Delta_n \rightarrow \text{trees} \rightarrow \text{NC} \rightarrow \text{CC}$



$\rightsquigarrow$



- color nodes opposite side according to incoming edge
- color edges with parent node's color

- if w subdivides Δ_{xyz} , connect w to highest of x, y, z on tree (so $d-c$, not $d-a$)

Now sort the tricolored tree left-to-right such that

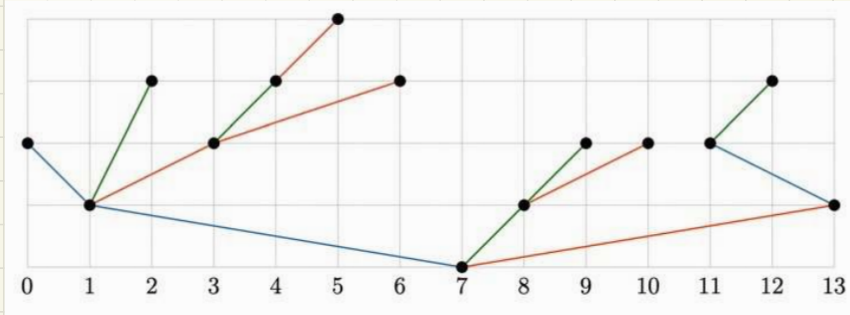
(1) $x \rightarrow y$ blue $\Rightarrow y$ & above left of x

(2) $x \rightarrow y$ green $\Rightarrow y$ & above right of x

(3) $x \rightarrow y$ red $\Rightarrow y$ & above right of x + right of all in green branch of x

$\Delta n \rightarrow \text{trees} \rightarrow \text{NC} \rightarrow \text{CC}$

E.g. An admissably ordered tricolored tree :



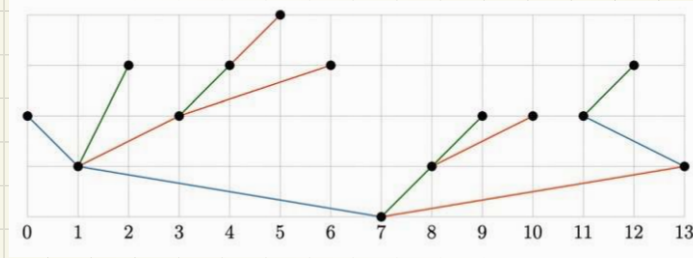
- Write $x \sim y$ when the path from x to y has no blue (just red/green),
 $x \rightarrow y$ when y is descended from x .
- Define $\pi_{R'}(x) := \max\{y \mid x \sim y\}$

$$\pi_R(x) := \max\left\{y \mid x \sim y, x \rightarrow y, \& \text{ either } x=y \text{ or the path from } x \text{ to } y \text{ begins with green}\right\}$$

E.g.

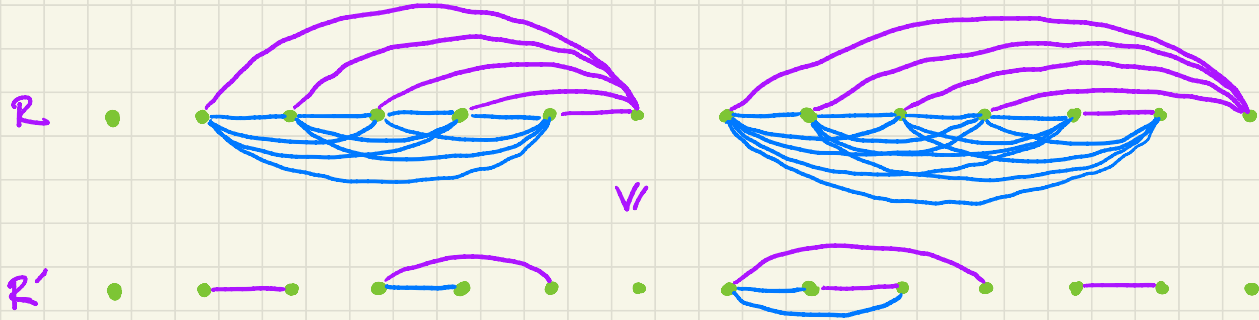
x	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\pi_{R'}(x)$	0	6	6	6	6	6	6	13	13	13	13	12	12	13
$\pi_R(x)$	0	2	2	5	4	5	6	10	9	9	10	12	12	13

$\Delta'n \rightarrow \text{trees} \rightarrow \text{NC} \rightarrow \text{CC}$



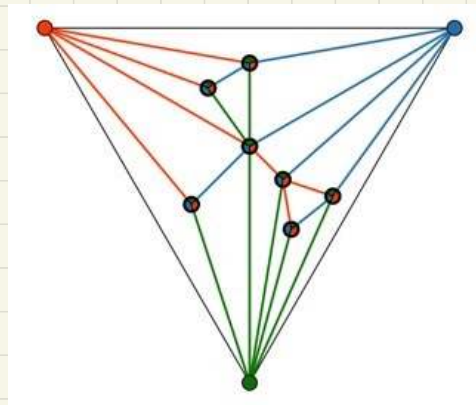
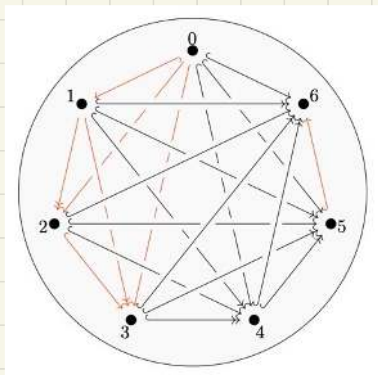
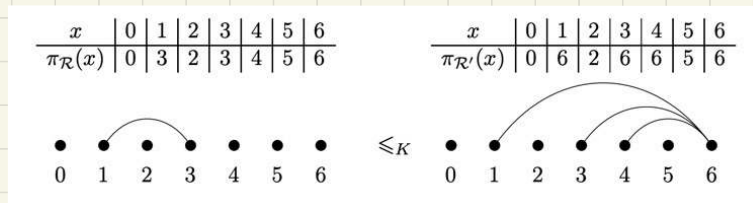
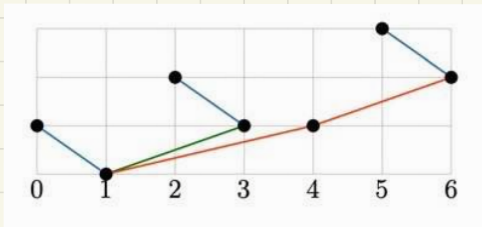
x	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\pi_{R'}(x)$	0	6	6	6	6	6	6	13	13	13	13	12	12	13
$\pi_R(x)$	0	2	2	5	4	5	6	10	9	9	10	12	12	13

(— = max'l
rel'ns $\leadsto$
NC part'n,
— = gen'l
rel'ns)



Model trees

Thm (BMO) Via these bijections, model structures on $[n]$ correspond to "blue-green trees growing from a red field." This recovers the enumeration of $Q([n])$ from **BOOR**.



Saturated Transfer Systems

A transfer system is saturated when it satisfies $2 \Rightarrow 3$.

Thm (Rubin) Transfer systems on $\text{Sub}(G)$ induced by G -linear isometries operads are saturated.

Thm (BMO) For a finite self-dual lattice P , the following structures are in bijective correspondence:

- saturated transfer systems,
- model str's in which all morphisms are fibrations,
- closure operators on P ,
- submonoids of $(P, \wedge)$, &
- monads on P .

Thm (Hafeez-Marcus-O-Osorno) Saturated transfer systems are generated by covering relations.

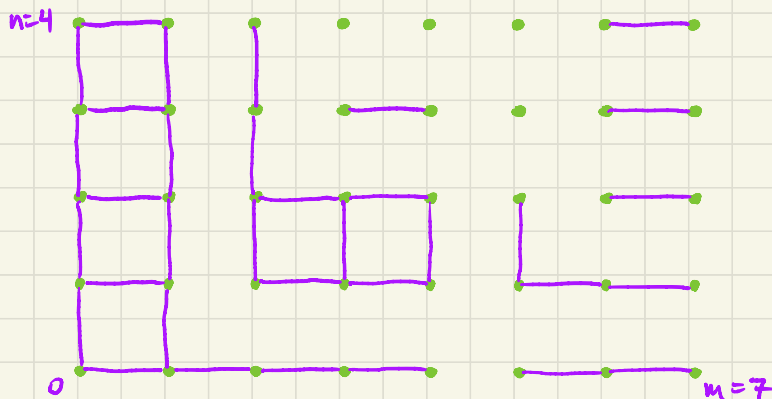
Matchsticks Again

Thm (HMOO) $T_r^{\text{sat}}([m] \times [n]) \cong \{ \text{legal matchstick configs on } [m] \times [n] \}$.

There are precisely $s(m, n) := \frac{1}{2} \sum_{j=2}^{m+2} (-1)^{m-j} \left\{ \begin{matrix} m+1 \\ j-1 \end{matrix} \right\} j! j^n$ of these, and

the exponential generating function for $s(m,n)$ takes the form

$$\sum_{m,n \geq 0} \frac{s(m,n)}{m!n!} x^m y^n = \frac{\exp(2x+2y)}{(\exp(x) + \exp(y) - \exp(x+y))^3}.$$



$n \backslash m$	0	1	2	3
0	1	2	4	8
1	2	7	23	78
2	4	23	115	533
3	8	73	533	3451

$\{k\}_2 =$ Stirling
number
of the second
kind $=$ #1-block
partitions of
a k -element set

$$s(7,4) = 58 \ 718 \ 873$$

Summary + Q's

On a finite lattice P ,

- $WFS(P) \cong Tr(P)$
- $Pre(P) \cong Int(Tr(P), \leq)$
- $CC(P) \cong Int(Tr(P), \leq)$
- $|WFS(P)| = Cat(n+1)$
- $|Pre[n]| = \frac{2}{(n+1)(n+2)} \binom{4n+5}{n}$
- $|CC([n])| = \frac{1}{2n+3} \binom{3n+3}{n+1}$
- $|Q([n])| = \binom{2n+1}{n}$
- Kreweras intervals, stacked Δ 's, & tricolored trees for $CC([n])$
- $Tr^{sat}(P) \cong \{(W, All, C) \in Q(P)\} \cong \{\text{closure operators on } P\} \cong \{\text{submonoids of } (P, \wedge)\}$
- $|Tr^{sat}([m] \times [n])| = s(m, n) = |\{\text{legal matchstick config's}\}|$

Questions • Are transfer systems on lattices already "out there"?

- Connections to generalized Catalan combinatorics / associahedra / cluster algebras / representation theory?
- $Q(P)$ for other families of lattices P ? $P = [1]^n$?

Thank you!

- Self-duality of the lattice of transfer systems via weak factorization systems, Franchere - O - Osorno - Qin - Waugh
- Model structures on finite total orders, Balchin - O - Osorno - Roitzheim
- Saturated and linear isometric transfer systems for cyclic groups of order $p^m q^n$, Hafeez - Marcus - O - Osorno
- Composition closed premodel structures and the Kreweras lattice, Balchin - MacBrough - O
- Lifting N_∞ operads from conjugacy data, Balchin - MacBrough - O
- The combinatorics of N_∞ operads for $C_{q p^n}$ and D_{p^n} , Balchin - MacBrough - O
- Access at kylormsby.github.io/research/.