ECHT REU

An online math REU organized by the Electronic Computational Homotopy Theory Research Community

June 12 - July 21, 2023
Fully Virtual
Minimal Prerequisites

Homotopical Combinatorics: Model structures and transfer systems on lattices

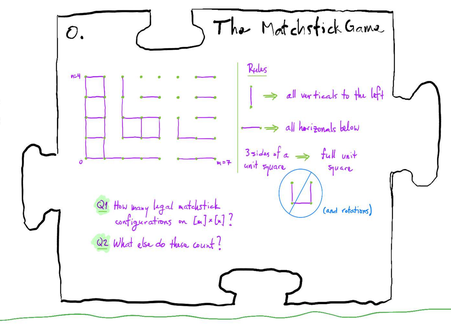
Program Details: s.wayne.edu/echt/reu-2023/
Apply: www.mathprograms.org/db/programs/1312

Kyle Ormsby | UW Combinatorics Seminar | 12.1.23

Homotopical Combinatorics

0. A puzzle
1. What is homotopy theory?
2. Weak factorization systems
3. Model structures
4. An invitation

0. The Matchstick Game



What is a homotopical context?

Category $\mathcal{C}$
$W, F, C \subseteq \mathrm{Mor}\,\mathcal{C}$
weak equivalences, fibrations, cofibrations

- Want to understand $\mathcal{C}[W^{-1}]$
- F, C + axioms provide control over localization

E.g.
- $\mathcal{C} = \mathrm{Top}$ or $\Delta^{op}\mathrm{Set}$, $W = \pi_*$-isos
- $\mathcal{C} = \mathrm{Ch}_{\geq 0}(R\text{-Mod})$, $W = $ qi-isos

$(\mathcal{C}, W, F, C)$ model category
model structure on $\mathcal{C}$

And why?

- Fibrant/cofibrant replacement mimics injective/projective resolution
- Bousfield localization: invert more morphisms, retain cofibrations
- Quillen adjunctions/equivalences compare model categories

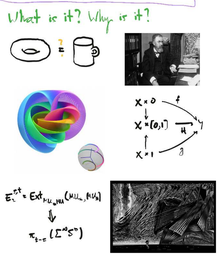
E.g. (Quillen, Sullivan, Bousfield-Guggenheim 1969-77)

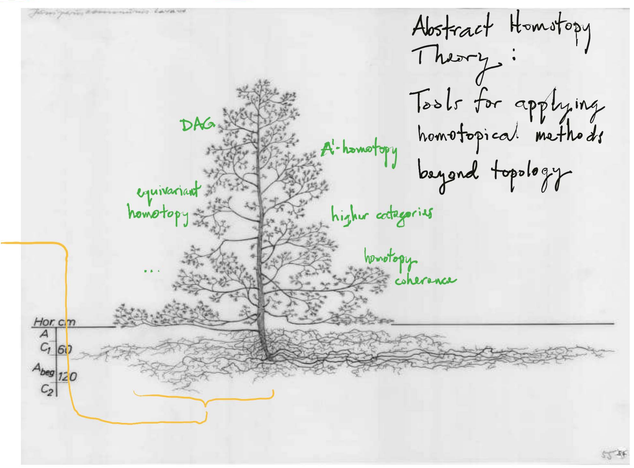
$$\begin{array}{ccc} \mathrm{Top} \simeq \Delta^{op}\mathrm{Set} & \rightleftarrows & \mathrm{dgca}_{\mathbb{Q}}^{\geq 0} \\ \cup & & \cup \\ \mathrm{Top}^{nft} & \underset{L\mathbb{Q}}{\rightleftarrows} & \mathrm{Top}^{nft}_{\mathbb{Q}} \end{array}$$

nft = nilpotent finite type

1. Homotopy Theory

What is it? Why is it?



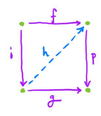


GOAL

Classify and count model structures on lattice posets.

- Poset $(P, \leq)$ is a set P with a transitive, reflexive, anti-symmetric relation $\leq$, corresponding category P has objects P, morphisms $P(x,y) = \begin{cases} * & \text{if } x \leq y \\ \emptyset & \text{o/w.} \end{cases}$
- P is a lattice if it admits all meets $\wedge$ = limits and joins $\vee$ = colimits

2. Weak Factorization Systems



Say i has the left lifting property wrt p (and p has the right lifting property wrt i) and write $i \boxtimes p$ when for $\forall f, g$ s.t. $\begin{smallmatrix} \cdot & \xrightarrow{f} & \cdot \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ \cdot & \xrightarrow{g} & \cdot \end{smallmatrix}$ commutes, $\exists\, h$ such that $f = hi$, $g = ph$.

For $M \subseteq \mathrm{Mor}\,\mathcal{C}$, write $^{\boxtimes}M := \{i \mid i \boxtimes p \ \forall p \in M\}$
$M^{\boxtimes} := \{p \mid i \boxtimes p \ \forall i \in M\}$.

A weak factorization system on $\mathcal{C}$ consists of $(L, R) \subseteq \mathrm{Mor}\,\mathcal{C} \times \mathrm{Mor}\,\mathcal{C}$ such that
WF1) $\mathrm{Mor}\,\mathcal{C} = R \circ L$ i.e. with $i \in L, p \in R$,
WF2) $L = {}^{\boxtimes}R$ i.e. $L \subseteq {}^{\boxtimes}R$ and $R \subseteq L^{\boxtimes}$, and
WF3) L, R closed under retracts.

Thm (Franchere-O-Osorno-Qin-Waugh)
If P is the category associated with a lattice, then R is the right set of a WFS on P iff
- R is a wide subcategory of P closed under pullbacks in P

or equivalently
- R is a subposet of P satisfying $x \to y \in R,\ z \in P \Rightarrow x \wedge z \to y \wedge z \in R$.

meets computed in P

We have $\mathrm{WFS}(P) = \{({}^{\boxtimes}R, R) \mid R \text{ as above}\}$.

Subposet: same set, potentially fewer relations (e.g. $(\mathbb{Z}, 1) \subseteq (\mathbb{Z}, \leq)$).

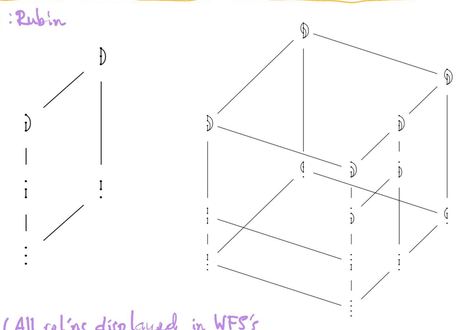
Ordering $\mathrm{WFS}(P)$ by relation inclusion results in a lattice.
Let $P = [n] = \{0 < 1 < \cdots < n\}$.

Thm (Balchin-Barnes-Roitzheim)
There is a lattice isomorphism
$\mathrm{WFS}([n]) \cong$ Tamari lattice $\mathbb{A}_{n+1}$ of planar full binary trees with $n+2$ leaves (under rotation $\curlyvee \to \curlyvee$).

As such, $|\mathrm{WFS}([n])| = \mathrm{Cat}(n+1) = \frac{1}{n+2}\binom{2n+2}{n+1}$.

(n+1)-st Catalan number

Images: Rubin



(All rel'ns displayed in WFS's, not just covering relations.)

3. Model structures

A premodel structure (Barton) on $\mathcal{C}$ is an interval of weak factorization systems on $\mathcal{C}$

$$\begin{array}{ccc} (\mathcal{C} & , & AF) \\ \cup & & \cap \\ (AC & , & F) \end{array}$$

cofibrations, $= {}^{\boxtimes}AF$; anodyne fibrations
anodyne cofibrations, $= {}^{\boxtimes}F$; fibrations

$\mathrm{Pre}(\mathcal{C}) = \mathrm{Int}(\mathrm{WFS}(\mathcal{C}))$ is the lattice of premodel structures on $\mathcal{C}$.

For finite chains we now have

$\mathrm{Pre}([n]) = \mathrm{Int}(\mathrm{WFS}([n])) \cong \mathrm{Int}(\mathbb{A}_{n+1})$.

Thm (Chapoton) $|\mathrm{Int}\,\mathbb{A}_n| = \frac{2}{n(n+1)}\binom{4n+1}{n-1}$

Cor (Balchin-O-Osorno-Roitzheim)
$|\mathrm{Pre}[n]| = \frac{2}{(n+1)(n+2)}\binom{4n+5}{n}$

Thm (Joyal-Tierney) For a bicomplete category $\mathcal{C}$,
$\{((C, AF), (AC, F)) \in \mathrm{Pre}(\mathcal{C}) \mid AF \circ AC \text{ satisfies } 2 \Rightarrow 3\}$
is in bijection with model structures on $\mathcal{C}$.

2 ⇒ 3: If two of f, g, gf belong to $AF \circ AC$, then so does the third.

The bijection is given by

$$\begin{array}{ccc} (C, AF) & & \\ \cup \quad \cap & \longmapsto & (AF \circ AC, F, C) \\ (AF, F) & & \end{array}$$

$$\begin{array}{ccc} (C, W \cap F) & & \\ \cup \quad \cap & \longmapsto & (W, F, C) \\ (W \cap F, F) & & \end{array}$$

Note Quillen's original definition of model structure used (redundant!) axioms MC1-MC5 on W, F, C.

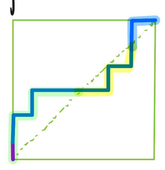
Thm (BOOR) The number of model structures on $[n]$ is $|\mathcal{Q}([n])| = \binom{2n+1}{n}$, and for $0 \leq k \leq n$, $\frac{2(k+1)}{n+k+2}\binom{2n+1}{n-k}$ of these have homotopy category $\cong [k]$.

Pf Idea Specify k s.t. $[k] \cong \mathrm{Ho}([n])$, then specify $k+1$ weak equivalence classes, then count choices of contractible model structures:

$$|\mathcal{Q}([n])| = \sum_{k=0}^{n} \sum_{\substack{i_0 + \cdots + i_k \\ = n+1}} \prod_{j=0}^{k+1} \mathrm{Cat}(i_j) = \binom{2n+1}{n}.$$

Shapiro (1976): $\frac{2(k+1)}{n+k+2}\binom{2n+1}{n-k}$

Final equality by lattice paths $(0,0)$ to $(n+1, n+1)$ with initial step north:
- $(i_0, \ldots, i_k)$ give diagonal x-ings
- Dyck paths have Catalan count

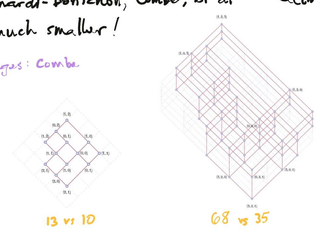


3'. Composition closed premodel structures

Fill the gap between $\mathrm{Pre}(P)$ & $\mathcal{Q}(P)$.

$\mathrm{Int}(\mathbb{A}_n)$ is well-studied by Chapoton, Bernardi-Bonichon, Combe, et al — $\mathcal{Q}([n-1])$ is much smaller!

Images: Combe



13 vs 10 | 68 vs 35

Call a premodel structure (C, AF) on $\mathcal{C}$ composition closed when (AC, F)

$W := AF \circ AC$ is closed under composition. (Need 2⇒3 for model str.)

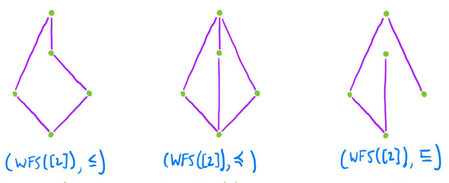
For $\mathcal{C} = P$ a lattice, write $R \subseteq R'$ for a premodel str / interval of transfer systems.

Thm (Balchin-MacBrough-O) For P a complete lattice, $\exists$ partial order $\trianglelefteq$ on $\mathrm{WFS}(P)$ refining $\leq$ and such that $R \trianglelefteq R'$ if and only if $R \circ {}^{\boxtimes}R'$ is closed under composition. Moreover, $R \trianglelefteq R'$ iff every square $\begin{smallmatrix} x & \to & z \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R'} \\ y & \to & w \end{smallmatrix}$ has a splitting $\begin{smallmatrix} x & \to & z' & \to & z \\ {\scriptstyle R}\downarrow & & {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R'} \\ y & \to & w' & \to & w \end{smallmatrix}$.

Thm (ct'd) If P is finite, then $(\mathrm{WFS}(P), \trianglelefteq)$ is a finite lattice. Thus
$\mathrm{CC}(P) = \mathrm{Int}(\mathrm{WFS}(P), \trianglelefteq)$ is a finite lattice.
composition closed premodel structures

Note There is also an ordering $\sqsubseteq$ on $\mathrm{WFS}(P)$ such that $R \sqsubseteq R'$ iff the pair forms a model structure, but $(\mathrm{WFS}(P), \sqsubseteq)$ is not a lattice.

E.g.



Specializing to $P = [n]$ again, $\trianglelefteq$ recovers the Kreweras order on noncrossing partitions.

For $R \in \mathrm{Tr}[n]$, define $\pi_R : [n] \to [n]$, $i \mapsto \max\{j \mid iRj\}$.

Prop (FOOQW) Write NC_{n+1} for the set of noncrossing partitions of $\{0, 1, \ldots, n\}$. Then $\mathrm{Tr}[n] \to \mathrm{NC}_{n+1}$, $R \mapsto \{\pi_R^{-1}\{i\} \mid 0 \leq i \leq n,\ \pi_R^{-1}\{i\} \neq \emptyset\}$ is a bijection.

$[n] = B_1 \sqcup \cdots \sqcup B_k$ is NC when $a, b \in B_i$, $x, y \in B_j$, and $a < x < b < y \Rightarrow i = j$.

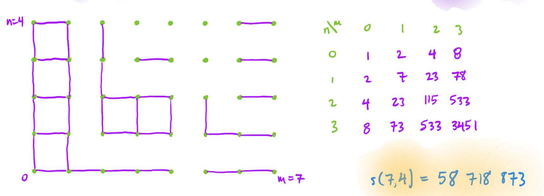
Thm (BMO) Pulling back the Kreweras (refinement) order on NC_{n+1} to $\mathrm{Tr}[n]$ gives the ordering $\trianglelefteq$.

Cor $\mathrm{CC}([n]) \cong \mathrm{Int}(\mathrm{NC}_{n+1})$ and thus
$|\mathrm{CC}([n])| = \frac{1}{2n+3}\binom{3n+3}{n+1}$.
Kreweras: $= |\mathrm{Int}(\mathrm{NC}_{n+1})|$

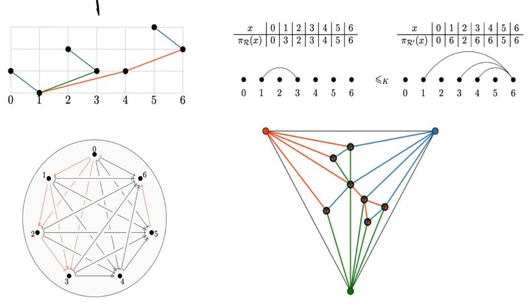
Has anyone solved the matchstick problem?

Legal matchstick configs on $[m] \times [n]$ correspond to saturated transfer systems and to model structures in which all morphisms are fibrations.

Thm (Hafeez-Marcus-O-Osorno) The number of such model structures on $[m] \times [n]$ is
$s(m,n) = \frac{1}{2}\sum_{j=2}^{m+2} (-1)^{m-j} \left\{ {m+1 \atop j-1} \right\} j! \cdot j^n$
with exponential generating function
$\sum_{m,n \geq 0} \frac{s(m,n)}{m!\,n!} x^m y^n = \frac{\exp(2x+2y)}{(\exp(x) + \exp(y) - \exp(x+y))^3}$.

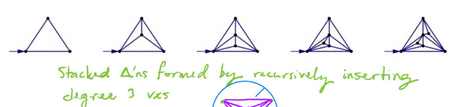


E.g. Via our bijections, the following structures all correspond to the same model structure



The following structures are equinumerous:
- Composition closed premodel structures on $[n]$
- ↕ BMO
- Kreweras intervals in NC_{n+1}
- ↕ Bernardi-Bonichon
- (admissably ordered) ternary trees on $n+1$ nodes
- ↕ BB
- stacked triangulations

} identify model str's among these



[BB]

Stacked Δ's formed by recursively inserting degree 3 vxs

not stacked

Thm (BMO) On a finite self-dual lattice P, the following structures are in bijective correspondence:
- saturated transfer systems,
- model structures with $F = All$,
- closure operators,
- closure systems,
- submonoids of $(P, \wedge)$,
- monads on P.

(R. Liu) For $P = [m] \times [n]$, this is half the number of $\lrcorner$-diagrams of shape $[m] \times [n]$.
(Also interpretable in terms of acyclic orientations of a complete bipartite graph, hyperplane arrangements, etc.)

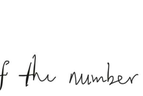
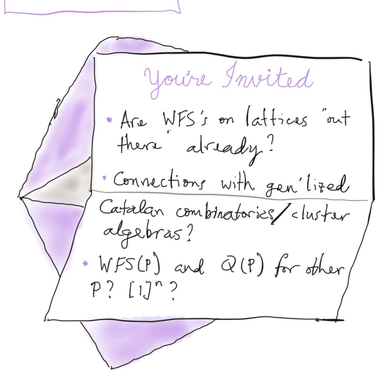


Image: Postnikov

4. An Invitation



And more: iterative constructions, power series, eDAG, etc.

References

- Self-duality of the lattice of transfer systems via weak factorization systems, Franchere-O-Osorno-Qin-Waugh
- Model structures on finite total orders, Balchin-O-Osorno-Roitzheim
- Saturated and linear isometric transfer systems for cyclic groups of order $p^n q^n$, Hafeez-Marcus-O-Osorno
- Composition closed premodel structures and the Kreweras lattice, Balchin-MacBrough-O
- Lifting N_∞ operads from conjugacy data, Balchin-MacBrough-O
- The combinatorics of N_∞ operads for C_{2p^n} and D_{p^n}, Balchin-MacBrough-O

- Access at kyleormsby.github.io/research/.