MATH 411: TOPICS IN ADVANCED ANALYSIS
HOMEWORK DUE WEDNESDAY WEEK 4

Problem 1. Suppose that a sequence (a_n) of real numbers is equidistributed mod 1, and that c is a real number. Show that $(a_n + c)$ is equidistributed mod 1.

Problem 2. Let $\phi = (1+\sqrt{5})/2$ and $\bar{\phi} = (1-\sqrt{5})/2$. Set $a_n = \phi^n + \bar{\phi}^n$.

(a) Prove that for $a_0 = 2$, $a_1 = 1$, and for $n \geq 1$,

$$a_{n+1} = a_n + a_{n-1}.$$

Deduce that $a_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$.

(b) Prove that

$$\langle \bar{\phi}^{2r+1} \rangle \to 0 \text{ as } \mathbb{N} \ni r \to \infty$$

and

$$\langle \bar{\phi}^{2r} \rangle \to 0 \text{ as } \mathbb{N} \ni r \to \infty.$$

(c) Deduce that

$$\lim_{n\to\infty} \frac{1}{n} \left|\{r \in \{1,\dots,n\} \mid \langle \phi^r \rangle \in [1/4, 3/4]\}\right| \to 0 \text{ as } n \to \infty$$

and thus (ϕ^r) is not equidistributed mod 1.

Problem 3. Let $B_\ell(x)$ denote the ℓ-th Bernoulli polynomial we defined in class.[1]

(a) Prove that $B_\ell(x)$ satisfies

$$B_\ell(1-x) = (-1)^\ell B_\ell(x).$$

In other words, B_ℓ is symmetric about $x = 1/2$ for ℓ even, and skew-symmetric about $x = 1/2$ for ℓ odd.

(b) Suppose that $B_{\ell-1}(x) = \sum_n c_n (x-1/2)^n$. Prove that

$$B_\ell(x) = \begin{cases} \sum_n \frac{c_n}{n+1}(x-1/2)^{n+1} - \sum_n \frac{c_n}{(n+1)(n+2)} \cdot \frac{1}{2^{n+1}} & \text{if } \ell \text{ is even,} \\ \sum_n \frac{c_n}{n+1}(x-1/2)^{n+1} & \text{if } \ell \text{ is odd.} \end{cases}$$

Problem 4 (continuation of Problem 3). We now connect $B_{2\ell}(1/2)$ to ζ-values.

(a) Making the same assumptions as we made in class, prove that

$$B_{2\ell}(1/2) = \frac{(-1)^\ell}{2^{\ell-1}\pi^{2\ell}} \sum_{n\geq 1} \frac{(-1)^{n+1}}{n^{2\ell}}.$$

(b) Prove that for $s \geq 2$ an integer,

$$\sum_{n\geq 1} \frac{(-1)^{n+1}}{n^s} = \zeta(s) - \frac{1}{2^{s-1}}\zeta(s)$$

and thus

$$B_{2\ell}(1/2) = \frac{(-1)^\ell}{2^{2\ell-1}\pi^{2\ell}} \left(1 - \frac{1}{2^{2\ell-1}}\right)\zeta(2\ell).$$

(c) Compute $B_3(x)$ as a polynomial in $(x - 1/2)$ and use Problem 3(b) to deduce the value of $B_4(1/2)$. Use this to compute $\zeta(4)$.

[1]Beware that many authors choose a different normalization of Bernoulli polynomials, for instance with $B_k'(x) = kB_{k-1}(x)$.

Problem 5. Let $\mathscr{H}$ and $\mathscr{H}'$ be Hilbert spaces, let $V \leq \mathscr{H}$ be a subspace of $\mathscr{H}$, and suppose that $T\colon V \to \mathscr{H}'$ is linear. Consider the following properties:

(UC) The operator T is uniformly continuous on V.

(C0) The operator T is continuous at $0 \in V$.

(B) The operator T is *bounded*, *i.e.*, there exists $M > 0$ such that for all $f \in V$,

$$\|T(f)\| \leq M\|f\|.$$

Note that condition (UC) implies (C0) *a fortiori*. Prove that all three conditions are equivalent by showing (C0) $\implies$ (B) $\implies$ (UC).