MATH 411: MIDTERM EXAM

Instructions. This is an open book, open notes, open Internet, closed collaboration, closed "AI" take-home exam. It is distributed Monday 10 March at 7A.M. and is due Wednesday 12 March at 10P.M. via Gradescope. The recommended exam period is 180 minutes, but this is only a suggestion. You are welcome — even encouraged — to read the problems early and then sit down to 'properly' write solutions later. The work that you turn in may be handwritten or typed, and all of it must be your own. You may use a computer and/or the Internet only for the following purposes:

- Accessing course materials from the course website or Zulip;
- Looking at reference texts or your own notes;
- Typesetting or looking up how to typeset something in LaTeX;
- Checking arithmetic and other computations with a calculator or computer algebra system.

The Honor Principle prevails.

Keep the following in mind as you work on the exam:

- **Unless explicitly instructed otherwise, you must show your work and justify your claims in order to receive full credit.**
- The solutions you submit should not contain scratch work. Do your scratch work on separate sheets of paper which you do not submit.

Your completed exam must be uploaded to Gradescope as a PDF by 10P.M. Pacific time on Wednesday 12 March. **Given the flexibility of this exam format, I reserve the right to not accept late submissions.** If something might prevent you from turning the exam in on time, please contact me ASAP.

Problems will be scored in a manner approximately proportional to the amount of work each requires. Exams will be returned via Gradescope, and I will supply exam solutions after all exams are graded upon request.

Problem 1. Suppose $f\colon \mathbb{R} \to \mathbb{C}$ is $1/2$-periodic, so that $f(x+1/2) = f(x)$ for all $x \in \mathbb{R}$. Show that $\hat{f}(n) = 0$ for all odd n.

Problem 2. Let $f\colon [a,b] \to \mathbb{C}$ be a continuous function. Show that for any $\varepsilon > 0$ there exists a polynomial $P \in \mathbb{C}[x]$ such that

$$\sup_{x\in[a,b]} |f(x) - P(x)| < \varepsilon$$

This is called the *Weierstrass approximation theorem.* [*Hint*: A corollary from the lecture of 5 February implies that continuous functions on S^1 may be uniformly approximated by trigonometric polynomials. You may want to also show that that $e^{2\pi i x}$ can be uniformly approximated by polynomials on any interval.]

Problem 3. Recall that for $s > 1$,

$$\zeta(s) := \sum_{n\geq 1} \frac{1}{n^s}.$$

(a) Fix a real number $t > 0$ and let

$$f_t(x) = \frac{t}{\pi(x^2 + t^2)}.$$

Prove that

$$\widehat{f_t}(\gamma) = e^{-2\pi t|\gamma|}.$$

(b) Prove that

$$\frac{1}{\pi}\sum_{n\in\mathbb{Z}} \frac{t}{t^2 + n^2} = \sum_{n\in\mathbb{Z}} e^{-2\pi t|n|}.$$

(c) Prove that the following identity holds for $0 < t < 1$:

$$\frac{1}{\pi}\sum_{n\in\mathbb{Z}} \frac{t}{t^2 + n^2} = \frac{1}{\pi t} + \frac{2}{\pi}\sum_{m\geq 1} (-1)^{m+1}\zeta(2m)t^{2m-1}.$$

(d) Prove that

$$\sum_{n\in\mathbb{Z}} e^{-2\pi t|n|} = \frac{2}{1 - e^{-2\pi t}} - 1.$$

(e) The *Bernoulli numbers* B_k are defined by the Taylor series for $z/(e^z - 1)$ in the sense that

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{m\geq 1} \frac{B_{2m}}{(2m)!} z^{2m}.$$

Use the above work to deduce that

$$2\zeta(2m) = (-1)^{m+1}\frac{(2\pi)^{2m}}{(2m)!}B_{2m}.$$

(You may *not* cite or use the results about ζ we proved in class or in homework.)