

Poisson summation as a trace

For $P \in \mathbb{R}^d$ compact with positive volume and

$$K: P \times P \longrightarrow \mathbb{C}$$

continuous, define

$$T_K: L^2(P) \longrightarrow L^2(P)$$

$$f \longmapsto \left(x \longmapsto \int_P K(x, y) f(y) dy \right)$$

$$\begin{aligned} &= \int_P \left(\int_P K(x, y) f(y) dy \right) \overline{f(x)} dx \\ &= \int_{P \times P} K(x, y) f(y) \overline{f(x)} dx dy \end{aligned}$$

Call K a kernel.

Check: T_K is linear

$$\begin{aligned} \langle T_K f, f \rangle &= \int_P T_K(f)(x) \cdot \overline{f(x)} dx \\ &= \int_P f(x) \cdot \overline{T_K f(x)} dx \\ \langle f, T_K f \rangle &= \int_P f(x) \cdot \overline{T_K f(x)} dx \end{aligned}$$

Recall T_K is positive when $\langle T_K f, f \rangle > 0$ for $f \neq 0$.

Check T_K is self-adjoint (aka Hermitian) when $K(x, y) = \overline{K(y, x)}$.

$$\langle T_K f, f \rangle = \langle f, T_K f \rangle$$

By the spectral theorem, T_K has an orthonormal basis of eigenvectors $\{v_1, v_2, \dots\}$ with $T_K v_i = \lambda_i v_i$.

Then [Mercer 1909] Suppose T_K is a positive self-adjoint operator on $P \subseteq \mathbb{R}^d$ compact. Then

$$K(x, y) = \sum_{n \geq 1} \lambda_n v_n(x) \overline{v_n(y)}$$

with absolute uniform conv. $\square$

Defn The trace of T_K is $\text{Tr}(T_K) := \sum_{n=1}^{\infty} \lambda_n$.

For T_K as in Mercer's thm, $(T_r(T) = \sum_{i \in \mathbb{N}} \langle T e_i, e_i \rangle$
(e_i) o.n. basis of $\mathcal{H}$)

$$\int_P K(x, x) dx = \int_P \sum_{n=1}^{\infty} \lambda_n v_n(x) \overline{v_n(x)} = \sum_{k=1}^{\infty} \lambda_n \int_P |v_n(x)|^2 dx = \sum_{k=1}^{\infty} \lambda_n$$

So in this case $\text{Tr}(T_K) = \int_P K(x, x) dx$ as well.

$$\text{so } \{ \mathbb{R}^d / \mathbb{Z}^d = \mathbb{T}^d$$

Set $P = [0, 1]^d$, $f: \mathbb{R}^d \rightarrow \mathbb{C}$ Schwartz.

Consider the linear operator

$$L_f : L^2([0,1]^d) \longrightarrow L^2(\mathbb{R}^d)$$

$$g \longmapsto (f * g : x \mapsto \int_{\mathbb{R}^d} f(x-y) g(y) dy)$$

↑ periodization
of g .

Compute

$$L_f(g)(x) = \int_{\mathbb{R}^d} f(x-y) g(y) dy$$

$$= \sum_{n \in \mathbb{Z}^d} \int_{[0,1]^d - n} f(x-y) g(y) dy$$

$$= \sum_{n \in \mathbb{Z}^d} \int_{[0,1]^d} f(x-y+n) g(y) dy$$

$$= g(y-n)$$

$$= \int_{[0,1]^d} \left(\sum_{n \in \mathbb{Z}^d} f(x-y+n) \right) g(y) dy$$

Set $K(x,y) = \sum_{n \in \mathbb{Z}^d} f(x-y+n)$. Then $L_f = T_K$.

Claim Eigenfunctions of L_f are $e_k : x \mapsto e^{2\pi i x \cdot k}$ for $k \in \mathbb{Z}^d$.

Check $L_f(e_k)(x) = \int_{\mathbb{R}^d} f(y) e^{2\pi i (x-y) \cdot k} dy$ ($f * g = g * f$)

$$= e^{2\pi i x \cdot k} \int_{\mathbb{R}^d} f(y) e^{-2\pi i y \cdot k} dy$$

$$= \hat{f}(k) e_k(x) \Rightarrow \hat{f}(k), k \in \mathbb{Z}^d \text{ are eigenvalues.}$$

Since the e_k are a basis for $L^2(\mathbb{T}^d)$,

$\{e_k \mid k \in \mathbb{Z}^d\}$ is an eigenbasis for L_f with associated eigenvalues $\{\hat{f}(k) \mid k \in \mathbb{Z}^d\}$.

$$\text{It follows that } \text{Tr}(L_f) = \sum_{k \in \mathbb{Z}^d} \hat{f}(k). \quad \begin{aligned} &\langle Tr_{v_i, v_i} \rangle \\ &= \lambda_i \langle v_i, v_i \rangle \\ &= \lambda_i \end{aligned}$$

Now suppose L_f is self-adjoint and positive.

$$\text{Then } \text{Tr}(L_f) = \int_{[0,1]^d} K(x,x) dx = \int_{[0,1]^d} \sum_{n \in \mathbb{Z}^d} f(n) dx$$

$$= \sum_{n \in \mathbb{Z}^d} f(n) \int_{[0,1]^d} dx \leftarrow$$

$$= \sum_{n \in \mathbb{Z}^d} f(n)$$

$$\text{So in this case, } \sum_{n \in \mathbb{Z}^d} f(n) = \sum_{k \in \mathbb{Z}^d} \hat{f}(k) \quad !$$

Note For $\mathbb{Z}^d$ replaced with $\mathcal{L} \subseteq \mathbb{R}^d$ full rank lattice
contributes $\det \mathcal{L}$

Q For which f is L_f positive?

Asking for $\langle f * g, g \rangle > 0$ for $g \neq 0$.

$$\begin{aligned}\langle f * g, g \rangle &= \int_{\mathbb{R}^d} (f * g)(x) \overline{g(x)} \, dx \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(x-y) g(y) \, dy \right) \overline{g(x)} \, dx\end{aligned}$$

This perspective for $\mathbb{Z}^d \leq \mathbb{R}^d$ generalizes of "finite discrete subgroup" $\Gamma \leq G \rightsquigarrow$ Selberg trace formula.