

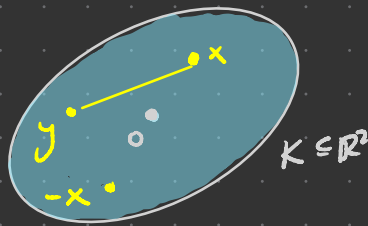
Minkowski's convex body theorem

Throughout, $K \subseteq \mathbb{R}^d$ is compact.

Call K centrally symmetric when

$$x \in K \iff -x \in K$$

K convex when $\forall x, y \in K$,
 $tx + (1-t)y \in K \quad \forall t \in [0, 1]$.



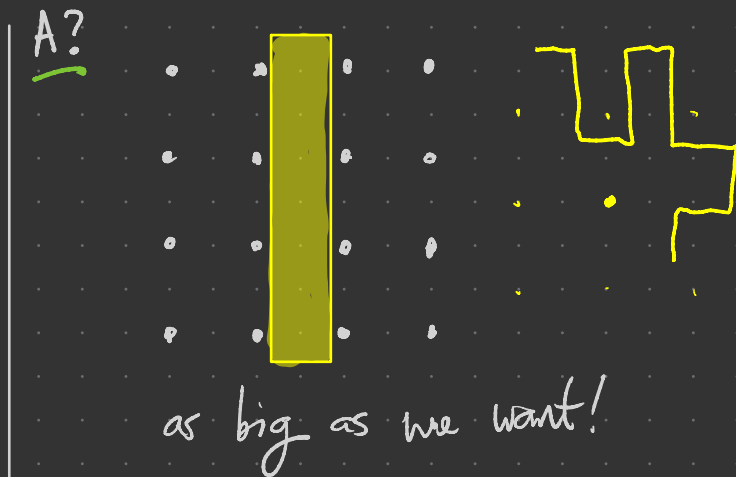
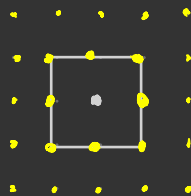
Say K contains a lattice point when $K \cap \mathbb{Z}^d \neq \emptyset$.

Motivating question How large must K be in order to contain a lattice point?

Thm [Minkowski] Let $K \subseteq \mathbb{R}^d$
 be compact, convex, and centrally
 symmetric. If $\text{vol } K > 2^d$, then
 $K^\circ \cap (\mathbb{Z}^d \setminus \{0\}) \neq \emptyset$

(i.e. the interior of K contains a
 nonzero lattice point).

Note The bound is sharp: $K = [-1, 1]^d$ has volume 2^d and
 $(-1, 1)^d \cap \mathbb{Z}^d = \{(0, 0)\}$.



Note Contrapositive: $K \subseteq \mathbb{R}^d$ compact convex centrally symmetric.
If $K \cap \mathbb{Z}^d = \{0\}$, then $\text{vol } K \leq 2^d$.

Minkowski's theorem follows from the following:

Thm [Siegel] $K \subseteq \mathbb{R}^d$ compact convex centrally symmetric.

If $K \cap \mathbb{Z}^d = \{0\}$, then

$$2^d = \text{vol } K + \frac{4^d}{\text{vol } K} \sum_{\xi \in \mathbb{Z}^d \setminus 0} |\hat{1}_{\frac{1}{2}K}(\xi)|^2$$

Here $rK := \{rx \mid x \in K\}$ for $r \in \mathbb{R}$.



$1_{\frac{1}{2}K} = \text{char fn of } \frac{1}{2}K$

There's even a more general version:

Thm [Siegel] Let $K \subseteq \mathbb{R}^d$ be compact and assume $1_{\frac{1}{2}K} * 1_{-\frac{1}{2}K}$ satisfies Poisson summation. If $(\frac{1}{2}K - \frac{1}{2}K)^\circ \cap \mathbb{Z}^d = \{0\}$, then

$$2^d = \text{vol } K + \frac{4^d}{\text{vol } K} \sum_{\xi \in \mathbb{Z}^d \setminus 0} \left| \hat{1}_{\frac{1}{2}K}(\xi) \right|^2.$$

Here $K+L = \{x+y \mid x \in K, y \in L\}$ and $\frac{1}{2}K - \frac{1}{2}K$ is the symmetrization of K .

Exercise Show $K-K$ is centrally symmetric.

Take $z \in K-K$. Then $z = x-y$, $x, y \in K$. Then

$-z = y-x$ and $y, x \in K$ so $-z \in K-K$.

Fact If K

is cent sym,
then

$$\frac{1}{2}K - \frac{1}{2}K = K.$$

Pf of Thm Set $f = 1_{\frac{1}{2}K} * 1_{-\frac{1}{2}K} \in C(\mathbb{R}^d)$. By Poisson summation,

$$\sum_{n \in \mathbb{Z}^d} f(n) = \sum_{\xi \in \mathbb{Z}^d} \hat{f}(\xi).$$

By definition of f ,

$$\begin{aligned} \sum_{n \in \mathbb{Z}^d} f(n) &= \sum_{n \in \mathbb{Z}^d} \int_{\mathbb{R}^d} 1_{\frac{1}{2}K}(y) 1_{-\frac{1}{2}K}(n-y) dy \\ &= \sum_{n \in \mathbb{Z}^d} \int_{\mathbb{R}^d} 1_{\frac{1}{2}K^0}(y) 1_{-\frac{1}{2}K^0}(n-y) dy. \end{aligned}$$

We have $y \in \frac{1}{2}K$ and $n-y \in -\frac{1}{2}K$ iff $n \in \frac{1}{2}K - \frac{1}{2}K$.

The only interior lattice point of $\frac{1}{2}K - \frac{1}{2}K$ is 0 by hypothesis. Thus only $n=0$ contributes:

$$\sum_{n \in \mathbb{Z}^d} f(n) = f(0) = \int_{\mathbb{R}^d} 1_{\frac{1}{2}K}(y) 1_{-\frac{1}{2}K}(-y) dy$$

$$= \int_{\mathbb{R}^d} 1_{\frac{1}{2}K}(y) dy$$

$$= \frac{\text{vol. } K}{2^d}.$$

Meanwhile,

$$\sum_{\xi \in \mathbb{Z}^d} \hat{f}(\xi) = \sum_{\xi \in \mathbb{Z}^d} \hat{1}_{\frac{1}{2}K}(\xi) \hat{1}_{-\frac{1}{2}K}(\xi)$$

$$f = 1_{\frac{1}{2}K} * 1_{-\frac{1}{2}K} \\ \Rightarrow \hat{f}(\xi) = \hat{1}_{\frac{1}{2}K}(\xi) \hat{1}_{-\frac{1}{2}K}(\xi)$$

$$= \sum_{\xi \in \mathbb{Z}^d} \int_{\frac{1}{2}K} e^{2\pi i \xi \cdot x} dx \int_{-\frac{1}{2}K} e^{2\pi i \xi \cdot x} dx$$

$$= \sum_{\xi \in \mathbb{Z}^d} \int_{\frac{1}{2}K} e^{2\pi i \xi \cdot x} dx \int_{\frac{1}{2}K} e^{2\pi i \xi \cdot (-x)} dx$$

$$= \sum_{\xi \in \mathbb{Z}^d} \int_{\frac{1}{2}K} e^{2\pi i \xi \cdot x} dx \int_{\frac{1}{2}K} e^{2\pi i \xi \cdot x} dx$$

$$= |\hat{1}_{\frac{1}{2}K}(0)|^2 + \sum_{\xi \in \mathbb{Z}^d \setminus 0} |\hat{1}_{\frac{1}{2}K}(\xi)|^2$$

$$= \left(\frac{\text{vol } K}{2^d} \right)^2 + \sum_{\xi \in \mathbb{Z}^d \setminus 0} \left| \hat{1}_{\frac{1}{2}K}(\xi) \right|^2$$

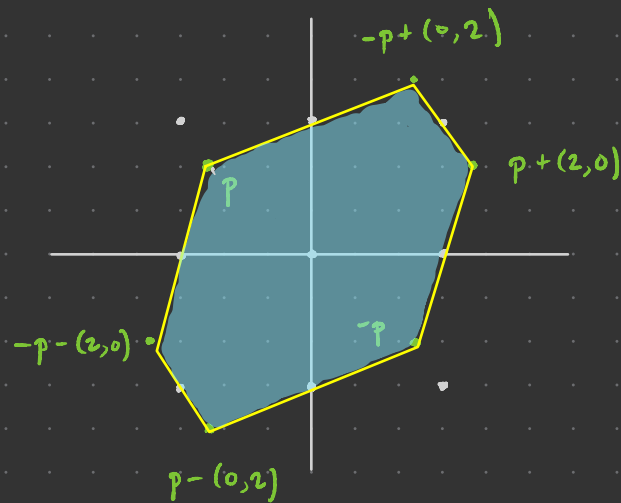
Hence
$$\frac{\text{vol } K}{2^d} = \left(\frac{\text{vol } K}{2^d} \right)^2 + \sum_{\xi \in \mathbb{Z}^d \setminus 0} \left| \hat{1}_{\frac{1}{2}K}(\xi) \right|^2$$

$$\Rightarrow 2^d = \text{vol } K + \frac{2^{2d}}{\text{vol } K} \sum_{\xi \in \mathbb{Z}^d \setminus 0} \left| \hat{1}_{\frac{1}{2}K}(\xi) \right|^2$$

□

Fact For $K \in \mathbb{R}^d$ compact convex, $1_K * 1_{-K}$ is "nice"
(satisfies Poisson summation).

E.g. For which $p \in \mathbb{R}^2$ does the following body have no nonzero integer points in its interior?



Where is area = 4?

There is also a version of Minkowski-Siegel for general full rank lattices in $\mathbb{R}^d$: $\mathcal{L} \subseteq \mathbb{R}^d$, $\mathcal{L} \cong \mathbb{Z}^d$ as Abelian groups

Thm Suppose $K \subseteq \mathbb{R}^d$ compact with $1_{\frac{1}{2}K} \neq 1_{-\frac{1}{2}K}$ "nice".

Let $\mathcal{L} \subseteq \mathbb{R}^d$ be a full rank lattice with dual lattice

$$\begin{aligned}\mathcal{L}^* &= \{x \in \mathbb{R}^d \mid x \cdot n \in \mathbb{Z} \text{ for all } n \in \mathcal{L}\} \\ &= M^{-T} \mathcal{L}.\end{aligned}$$

$$\left. \begin{aligned}(\mathbb{Z}^d)^* \\ = \mathbb{Z}^d\end{aligned} \right\}$$

If $(\frac{1}{2}K - \frac{1}{2}K)^0 \cap \mathcal{L} = \{0\}$, then

$$2^d \det \mathcal{L} = \text{vol } K + \frac{4^d}{\text{vol } K} \sum_{\xi \in \mathcal{L}^* \setminus 0} |\hat{1}_{\frac{1}{2}K}(\xi)|^2.$$

$\uparrow$
 $\det M$, M lin trans $\mathbb{R}^d \rightarrow \mathbb{R}^d$ s.t. $M(\mathbb{Z}^d) = \mathcal{L}$.

Note

$$f \in L^1(\mathbb{R}^d)$$

$$M: \mathbb{R}^d \rightarrow \mathbb{R}^d \quad \text{linear}$$

$$\widehat{f \circ M} \quad \text{involves} \quad M^{-T}$$