

Fourier analysis on $\mathbb{R}^d$

Recall $f \in L^1(A)$, $\hat{f}: \hat{A} \longrightarrow \mathbb{C}$

$$x \longmapsto \int_A f(x) \overline{x(\kappa)} dx$$

For $A = \mathbb{R}^d$, $\hat{A} \cong \mathbb{R}^d$ via $\xi \in \mathbb{R}^d \longmapsto \left(\mathbb{R}^d \longrightarrow \mathbb{C} \right)$

$$x \longmapsto e^{2\pi i \xi \cdot x}$$

We may thus rewrite $\hat{f}: \mathbb{R}^d \longrightarrow \mathbb{C}$

$$\xi \longmapsto \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x} dx$$

standard
dot product

Lemma If $f \in L^1(\mathbb{R}^d)$, then (a) $\|\hat{f}\|_\infty \leq \|f\|_1$, and

(b) $\hat{f}$ is uniformly continuous.

PF For (a), given $\xi \in \mathbb{R}^d$, we have

$$|\hat{f}(\xi)| = \left| \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x} dx \right|$$

$$\leq \int_{\mathbb{R}^d} |f(x) e^{-2\pi i \xi \cdot x}| dx$$

$$= \int_{\mathbb{R}^d} |f(x)| dx$$

$$= \|f\|_1.$$

For (b), fix $\xi, h \in \mathbb{R}^d$. Then

$$\left| \hat{f}(\xi+h) - \hat{f}(\xi) \right| = \left| \int_{\mathbb{R}^d} f(x) \left(e^{-2\pi i (\xi+h) \cdot x} - e^{-2\pi i \xi \cdot x} \right) dx \right|$$

$$= \left| \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x} \left(e^{-2\pi i h \cdot x} - 1 \right) dx \right|$$

$$\leq \int_{\mathbb{R}^d} |f(x)| \left| e^{-2\pi i h \cdot x} - 1 \right| dx$$

Let $g_h(x) := f(x)(e^{-2\pi i h \cdot x} - 1)$. Then $|g_h(x)| \leq 2|f(x)|$

and $\lim_{h \rightarrow 0} g_h(x) = 0$. Since g_h dominated by L^1 function $2f$,

we can use dominated convergence to conclude

$$\lim_{h \rightarrow 0} \int_{\mathbb{R}^d} |g_h(x)| dx = \int_{\mathbb{R}^d} \left(\lim_{h \rightarrow 0} |g_h(x)| \right) dx = 0$$

Thus $|\hat{f}(\xi+h) - \hat{f}(\xi)| \xrightarrow{h \rightarrow 0} 0$ uniformly in ξ . $\square$

E.g. Take $S \subseteq \mathbb{R}^d$ bounded and measurable. Then $1_S \in L^1(\mathbb{R}^d)$

$$\text{and } |\hat{1}_S(\xi)| \leq \|1_S\|_1 = \int_S dx = \mu(S)$$

$\uparrow$
characteristic
function of S

2 $\hat{1}_S \notin L^1(\mathbb{R}^d)$ in general.

pf Suppose for contradiction $\hat{1}_S \in L^1(\mathbb{R}^d)$. Then

$$\hat{\hat{1}}_S(x) = 1_S(-x)$$

is continuous, but of course 1_S is not! $\square$

E.g. If $S = \prod_{i=1}^d S_i$ then $\hat{1}_S(\xi) = \prod_{i=1}^d \hat{1}_{S_i}(\xi_i)$ by $\hat{1}_S = \int_S e^{-2\pi i \xi \cdot x} dx$

Recall $\hat{1}_{[-\frac{1}{2}, \frac{1}{2}]}(\xi) = \frac{\sin(\pi \xi)}{\pi \xi} = \text{sinc}(\xi)$

$$\text{so } \hat{1}_{\prod_{k=1}^d [-\frac{1}{2}, \frac{1}{2}]}(\xi) = \frac{1}{\pi^d} \prod_{k=1}^d \frac{\sin(\pi \xi_k)}{\xi_k}$$

$$= \int_S e^{-2\pi i (\xi_1 x_1 + \dots + \xi_d x_d)} dx$$

$$= \int \prod_k e^{-2\pi i \xi_k x_k} dx$$

Parlor trick Since $\hat{f}(x) = f(-x)$, we have $\hat{\hat{f}}(0) = f(0)$

$$\begin{aligned} \text{i.e. } f(0) &= \int_{\mathbb{R}^d} \hat{f}(\xi) \underbrace{e^{-2\pi i \xi \cdot 0}}_1 d\xi \\ &= \int_{\mathbb{R}^d} \hat{f}(\xi) d\xi. \end{aligned}$$

For $f = 1_{[-1/2, 1/2]^d}$, this says

$$1 = \int_{\mathbb{R}^d} \prod_{i=1}^d \frac{\sin(\pi \xi_i)}{\pi \xi_i} d\xi$$

Multivariable Poisson summation For "nice" $f \in L^1(\mathbb{R}^d)$,

$$\sum_{n \in \mathbb{Z}^d} f(n+x) = \sum_{\xi \in \mathbb{Z}^d} \hat{f}(\xi) e^{2\pi i \xi \cdot x} \quad \text{for all } x \in \mathbb{R}^d.$$

In particular,
($x=0$) $\sum_{n \in \mathbb{Z}^d} f(n) = \sum_{\xi \in \mathbb{Z}^d} \hat{f}(\xi).$

Idea Look at $f = 1_S$ for $S \subseteq \mathbb{R}^d$.

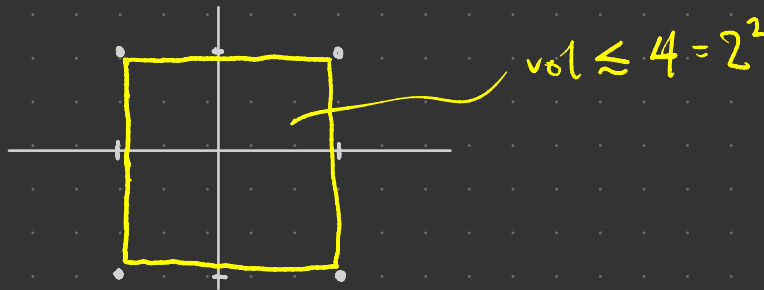
$$\text{Then } \sum_{n \in \mathbb{Z}^d} 1_S(n) = \# \{n \in \mathbb{Z}^d \mid n \in S\}.$$

Challenge: count on the Fourier side as well!

Minkowski's Thm (baby version)

Suppose $C \subseteq \mathbb{R}^d$ is a compact ^{convex!} subset that is centrally symmetric ($x \in C \Rightarrow -x \in C$). If 0 is the only lattice point ($\mathbb{Z}^d$) in $C \setminus \partial C$, then

$$\text{vol}(C) \leq 2^d$$



$$\text{Siegel: } 2^d - \text{vol}(C) = ?$$

$\underbrace{\hspace{1cm}}$
 Siegel tells us what
 this is!

$$\frac{1}{2}K - \frac{1}{2}K = \left\{ \frac{1}{2}(x-y) \mid x, y \in K \right\}$$

↳ symmetrization