

G, H LC groups then $G \times H$ is an LC group

Thm Let $I_G : C_c(G) \rightarrow \mathbb{C}$ be a Haar integral on G .
 $f \mapsto \int_G f$

Then $\forall f \in C_c(G \times H)$, the function

$$H \longrightarrow \mathbb{C}$$

$$y \longmapsto I_G(f(-, y)) = \int_G f(x, y) dx$$

lies in $C_c(H)$. Let $I_H = \int_H$ be a Haar integral on H .

Then a Haar integral on $G \times H$ is given by

$$I(f) = \int_H \int_G f(x, y) dx dy \quad \text{iterated integral} \\ = \text{Haar integral}$$

This can be performed in opposite order, yielding the same result, so

$$\int_H \int_G f(x,y) dx dy = \int_G \int_H f(x,y) dy dx.$$

Proof Endow $C_c(G)$ with topology of uniform convergence:

$f_n \xrightarrow{n \rightarrow \infty} f$ means f_n conv unif to f and $\exists K \subseteq G$ compact
with $\text{supp}(f_n) \subseteq K \quad \forall n$.
 $\overset{C_c(G)}{\text{with}} \quad \overset{C_c(G)}{\text{with}}$

Lemma $I_G: C_c(G) \rightarrow \mathbb{C}$ is cts ($f_n \rightarrow f$ in $C_c(G) \Rightarrow I(f_n) \rightarrow I(f)$ in $\mathbb{C}$).

7f Take $f_n \rightarrow f$, $K \in G$ as above. Let $\chi \in C_c^+(G)$ satisfy $\chi|_K = 1$.

Define $c := \int_G \chi$. Fix $\varepsilon > 0$ and take $N \in \mathbb{N}$ s.t. $n \geq N$

implies $|f_n(x) - f(x)| < \frac{\varepsilon}{c} \quad \forall x \in G$. Then for $n \geq N$,

$$\left| \int_G f_n - \int_G f \right| = \left| \int_G (f_n - f) \right|$$

$$\leq \int_G |f_n - f|$$

$$= \int_G \chi(x) |f_n(x) - f(x)| dx$$

$$< \frac{\varepsilon}{c} \int_G \chi = \varepsilon. \quad \square$$

Lemma Fix $f \in C_c(G \times H)$. Then the function

$$\begin{aligned} H &\longrightarrow \mathbb{C} \\ y &\longmapsto \int_G f(x, y) dx \end{aligned}$$

is in $C_c(H)$.

pf Suppose $y_n \rightarrow y$ in H , let $f_n(x) = f(x, y_n)$. By uniform continuity of f (why?)[✓], $f_n \rightarrow f(-, y)$ in $C_c(G)$.

Thus $y \mapsto \int_G f(-, y)$ is cts. $\parallel f(-, y_n)$

The projection $G \times H \xrightarrow{p_2} H$ is cts, so $p_2(\text{supp}(f)) \subseteq H$ is compact.

Thus $y \mapsto \int_G f(-, y)$ has compact support. $\square$

Now define $I_1(f) := \int_H \int_G f(x, y) dx dy$.

Lemma I_1 is a Haar integral on $G \times H$.

Pf For $s = (x_0, y_0) \in G \times H$,

$$I_1(L_s f) = \int_H \int_G f(x_0^{-1} x, y_0^{-1} y) dx dy$$

$$= \int_H \int_G f(x, y_0^{-1} y) dx dy \quad [\int_G \text{ Haar}]$$

$$= \int_H \int_G f(x, y) dx dy \quad [\int_H \text{ Haar}]$$

$$= I_1(f) \quad \square$$

Similarly, $I_2(f) = \int_G \int_H f(x, y) dy dx$ defines a Haar integral on $G \times H$. Thus $I_1(f) = c I_2(f)$ for some $c \geq 0$.

If $f(x, y) = g(x)h(y)$ for $g \in C_c^+(G)$, $h \in C_c^+(H)$, then

$$f \in C_c^+(G \times H) \quad \text{and} \quad I_1(f) = \int_G g(x) dx \cdot \int_H h(y) dy = I_2(f),$$

so $c = 1$. $\square$

Let's extend to more functions:

$f: G \rightarrow \mathbb{R}_{\geq 0}$ cts. Define no assumption on support.

$$\int_G f := \sup_{\substack{\varphi \in C_c(G) \\ 0 \leq \varphi \leq f}} \int_G \varphi \in [0, \infty].$$

Define $L^1_{bc}(G) := \{ f: G \rightarrow \mathbb{C} \mid f \text{ bdd, cts, } \|f\|_1 = \int_G |f| < \infty \},$

∩

$$L^2_{bc}(G) := \{ f: G \rightarrow \mathbb{C} \mid f \text{ bdd, cts, } \|f\|_2^2 = \int_G |f|^2 < \infty \}.$$

For $f \in L^1_{bc}(G)$, write $f = u + iv$ for real-valued $u, v \in L^1_{bc}(G)$.

Then we can define

$$\int_G f = \int_G \overbrace{u_+}^{\max\{u, 0\}} - \int_G \underbrace{u_-}_{\max\{-u, 0\}} + i \left(\int_G v_+ - \int_G v_- \right).$$

Then [Haar Fubini] Let $f \in L^1_{bc}(G \times H)$. Then $y \mapsto \int_G f(-, y)$ is in $L^1_{bc}(H)$ and $x \mapsto \int_H f(x, -)$ is in $L^1_{bc}(G)$.

We have
$$\int_G \int_H f(x, y) dy dx = \int_H \int_G f(x, y) dx dy. \quad \square$$

Note $L^2_{bc}(G)$ with $\langle f, g \rangle = \int_G f \cdot \bar{g}$ is an inner product space with Hilbert space completion $L^2(G) =$ Hilbert space completion of $C_c(G)$

Fourier transforms for functions on LCA groups

A an LCA group, $\int_A$ a Haar integral

$$\hat{A} = \{ \underset{\text{unitary}}{\text{cts characters}} \chi: A \rightarrow S^1 \}$$

For $f \in L^1_{bc}(A)$, define

$$\begin{aligned} \hat{f}: \hat{A} &\longrightarrow \mathbb{C} \\ \chi &\longmapsto \int_A f(x) \overline{\chi(x)} dx = \langle f, \chi \rangle \end{aligned}$$

✓ true for $f \in L^2$, A compact

Explore What does this look like for $A = S^1$, $A = \mathbb{R}$?

$$A = S' \Rightarrow \hat{A} \cong \mathbb{Z}$$

$$\hat{f} : \mathbb{Z} \longrightarrow \mathbb{C}$$

$$k \longmapsto \int_{S'} f(x) e^{-2\pi i k x} dx = \hat{f}(k)$$

character

$$x \mapsto e^{2\pi i k x}$$