

Haar integration

Context : locally compact Hausdorff groups (LC groups)

E.g. • $G = GL_n(\mathbb{R})$ as a subspace of $\mathbb{R}^{n \times n} \cong \mathbb{R}^{n^2}$

- G an LCA group
- G any Lie group
- $GL_n(\mathbb{Q}_p)$
- Bohr compactification

Defn $C_c(G) = \{f: G \rightarrow \mathbb{C} \mid f \text{ cts with compact support}\}$

here $\text{supp}(f) = \overline{\{x \in G \mid f(x) \neq 0\}}$

Call $f \in C_c(G)$ nonnegative ($f \geq 0$) when $\forall x \in G, f(x) \in \mathbb{R}_{\geq 0}$.

A linear functional $I: C_c(G) \rightarrow \mathbb{C}$ is an "integral" if

$$f \geq 0 \Rightarrow I(f) \geq 0.$$

E.g. • Riemann integral $\int_{-\infty}^{\infty} (\cdot) dx$ on $(\mathbb{R}, +)$

• $\delta_x: C_c(G) \rightarrow \mathbb{C}$ the Dirac distribution at x .
 $f \mapsto \delta_x(f) = f(x)$

Fix an integral $I: C_c(G) \rightarrow \mathbb{C}$ and write $I(f) =: \int_G f(x) dx$.

Lemma $\left| \int_G f(x) dx \right| \leq \int_G |f(x)| dx \quad \star$ $= \int_G f$

Pf Reduction to real-valued $f \in C_c(G)$:

First, if $f: G \rightarrow \mathbb{R} \in C_c(G)$, then $\int_G f \in \mathbb{R}$ (why?)

Thus $\text{Re}\left(\int_G f\right) = \int_G \text{Re}(f)$.
for gen'l f .

$$f_+(x) = \max\{f(x), 0\} \Rightarrow f = f_+ - f_-$$
$$f_-(x) = \max\{-f(x), 0\}$$

For $\theta \in \mathbb{S}^1$, $\int_G \theta f = \theta \int_G f$ so multiplying f by θ doesn't change either side of $\star$. So WLOG, $\int_G f \in \mathbb{R}$.

Now suppose we have proven $\star$ for f real-valued.

Then $\left|\int_G f\right| = \left|\text{Re}\left(\int_G f\right)\right| = \left|\int_G \text{Re}(f)\right| \leq \int_G |\text{Re}(f)| \leq \int_G |f|$
for gen'l f , $\star$ $|\text{Re}(f)| \leq |f|$

Now to prove $\oplus$ for real-valued f :

Let $f_{\pm} := \max\{\pm f, 0\}$. Then $f_{\pm} \in C_c(G)$, $f_{\pm} \geq 0$,

and $f = f_+ - f_-$, so that

$$\left| \int_G f \right| = \left| \int_G f_+ - \int_G f_- \right|$$

$$\leq \left| \int_G f_+ \right| + \left| \int_G f_- \right|$$

$$= \int_G f_+ + \int_G f_- \quad (\text{nonnegative})$$

$$= \int_G |f| \quad \square \quad [f_+ + f_- = |f|]$$

We now seek a special type of integral on G that plays well with translation:

Defn For $g \in G$, $f \in C_c(G)$, define

$$\begin{aligned} L_g f : G &\longrightarrow \mathbb{C} \\ x &\longmapsto f(g^{-1}x) \end{aligned}$$

the left translation of f by g .

Note $L_g f \in C_c(G)$ as well, and

$$L_g(L_h f) = L_{gh} f, \quad L_e f = f$$

Q Why $f(g^{-1}x)$ and not $f(gx)$?

• Want $(L_g f)(g) = f(e)$

• $(L_a f)(x) = f(-a+x)$

for $G = (\mathbb{R}, +)$

so this is a left action of G on $C_c(G)$.

or Haar

Defn An integral $I: C_c(G) \rightarrow \mathbb{C}$ is (left) invariant

when $I(L_g f) = I(f) \quad \forall f \in C_c(G), g \in G.$

(Equivalently, $\int_G f(gx) dx = \int_G f(x) dx \quad \forall g \in G, f \in C_c(G).$)

Eg. The Riemann integral $\int_{-\infty}^{\infty} f(x) dx$ is left invariant for $(\mathbb{R}, +)$.

Exercise Show that $f \mapsto \int_0^{\infty} \frac{f(x)}{x} dx$ is a Haar integral for $(\mathbb{R}_{>0}, \cdot)$.

Need: • linearity • nonnegativity • left-invariance

For left-invariance, $a > 0$

$$L_a f \mapsto \int_0^\infty \frac{f(a^{-1}x)}{x} dx = \int_0^\infty \frac{f(u)}{u} du$$

$u = a^{-1}x$
 $du = a^{-1} dx$

$$= \int_0^\infty \frac{f(u)}{u} du$$

✓

Thm There exists a nontrivial Haar integral I for G .

If I' is a second Haar integral, then $\exists c \geq 0$ s.t. $I' = cI$.

Pf Appendix B of Deitmar. □

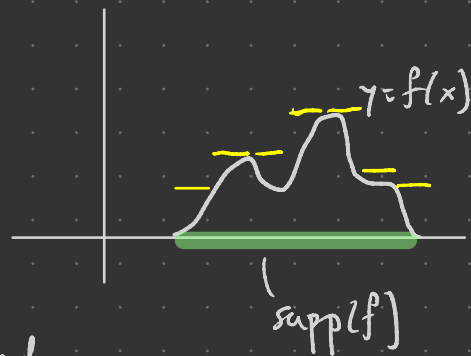
Construction

First recall the Riemann integral for $f: \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ cts, cptly supp'd.

For $n \in \mathbb{N}_{\geq 1}$, let $\chi_n := \chi_{[-\frac{1}{2n}, \frac{1}{2n}]}$. There exist $x_1, \dots, x_m \in \mathbb{R}$

$c_1, \dots, c_m > 0$ such that

$$f(x) \leq \sum_{j=1}^m c_j \chi_n(x - x_j)$$



Define $(f: \chi_n) := \inf \left\{ \sum_{j=1}^m c_j \mid \begin{array}{l} c_1, \dots, c_m > 0 \text{ and} \\ \exists x_1, \dots, x_m \in \mathbb{R} \text{ s.t. } f(x) \leq \sum_{j=1}^m c_j \chi_n(x - x_j) \end{array} \right\}$

$$\text{Then } \int_{-\infty}^{\infty} f(x) dx = \lim_{n \rightarrow \infty} \frac{1}{n} (f: \chi_n).$$

Note that if $f_0 = \chi_{[0,1]}$, then $(f_0 : \chi_n) = n$ and

$$\int_{-\infty}^{\infty} f(x) dx = \lim_{n \rightarrow \infty} \frac{(f : \chi_n)}{(f_0 : \chi_n)}.$$

This generalizes: Fix $f_0 : G \rightarrow \mathbb{R}_{\geq 0}$ nonzero.

$$\text{Set } \int_G f(x) dx = \lim_{u \rightarrow \{e\}} \frac{(f : \chi_u)}{(f_0 : \chi_u)} \quad \text{for } f : G \rightarrow \mathbb{R}_{\geq 0} \in C_c(G)$$

where: u = open nbhd of $e \in G$ shrinking to $\{e\}$.

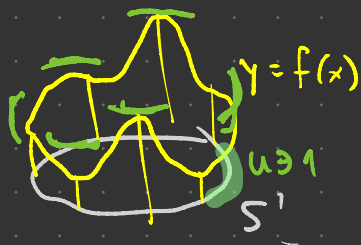
$$(f: \chi_u) := \inf \left\{ \sum_{j=1}^m c_j \mid c_1, \dots, c_m > 0 \text{ and } \exists g_1, \dots, g_m \in G \text{ s.t. } f(x) \leq \sum_{j=1}^m c_j L_{g_j} \chi_u(x) \right\}$$

Fact This is a nontrivial left-inv't integral.

Fact $(C_c(G), \langle, \rangle)$ is an inner product space with

$$\langle f_0, f_1 \rangle = \int_G f_0 \cdot \bar{f}_1. \quad (\text{for } \int_G \text{ Haar integration})$$

Defn The Hilbert space completion of $C_c(G)$ is called $L^2(G)$.



(f, x_u) optimizes trans's, heights
to make this small

$\lim_{u \rightarrow \text{leaf}} F(u) = L$ means

$\forall \epsilon > 0 \exists U$ nbhd of c s.t.

$$|F(u) - L| < \epsilon.$$