

Henceforth, A is a locally compact Hausdorff topological group;
shorthand LCA group (the H is silent)

Defn The Pontryagin dual of A is

$$\hat{A} := \underset{\text{cts}}{\text{Hom}}(A, S')$$

$$\underset{V'}{\mathbb{C}}^{\times} = \underset{V'}{\text{GL}}_1(\mathbb{C})$$

$$S' = \underset{V'}{U}_1(\mathbb{C})$$

the group of unitary characters of A under pointwise multiplication.

We endow $\hat{A}$ with the compact open topology: the coarsest topology with opens $P(K, U) := \{\chi \in \hat{A} \mid \chi K \subseteq U\}$

for all $K \subseteq A$ compact, $U \subseteq S'$ open.

Note Opens of $\hat{A}$ = unions of finite intersections of $P(K_i, U_i)$.

E.g. $\hat{\mathbb{Z}} \cong \mathbb{Z}$ w/ discrete topology

$\hat{\mathbb{R}} \cong \mathbb{R}$ w/ standard topology

$\hat{A} \cong A$ for A finite discrete.

Prop If A is an LCA group, then $\hat{A}$ is an LCA group.

Pf ^{For local compactness} Suffices to show $1 \in \hat{A}$ has a compact neighborhood.

Let K be a compact nbhd of e in A ,

$$U := e^{2\pi i(-1/4, 1/4)}$$

Claim $\overrightarrow{P(K, U)}$ is a compact nbhd of $1 \in \hat{A}$.

PF HW.

Why is $\hat{A}$ Hausdorff?

Suffices to show $\{1\} \subseteq \hat{A}$ closed

$\Leftrightarrow \hat{A} \setminus 1$ open. But $x \neq 1 \Rightarrow x(a) \neq 1$ for some $a \in A \setminus e$.

Let $U = S^1 \setminus 1$. Then $x \in P(\{a\}, U)$ but $1 \notin P(\{a\}, U)$.

Thus $\hat{A} \setminus 1 = \bigcup_{a \neq e} P(\{a\}, U)$ is open, as desired.

Finally, we need to show multiplication, inversion are continuous.

We can do this all together by proving $f: \hat{A} \times \hat{A} \rightarrow \hat{A}$
 $(x, y) \mapsto xy^{-1}$

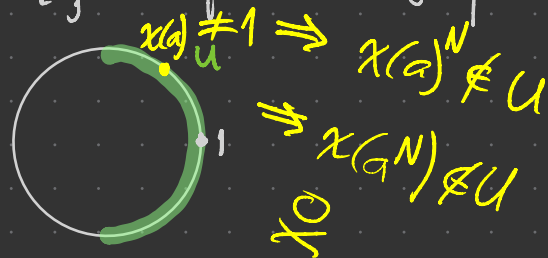
is cts. (Why?) Moral ex: can be checked
 by showing $x_i, y_i \xrightarrow{\text{loc unif}} x, y \Rightarrow x_i y_i^{-1} \xrightarrow{\text{loc unif}} xy^{-1}$

Thm For A an LCA group, A discrete $\Rightarrow \hat{A}$ compact
 A compact $\Rightarrow \hat{A}$ discrete.
 every subset is open $\Leftrightarrow$ all singletons open

Pf First suppose A compact. Then $\forall U \in \mathcal{S}'$ open, $P(A, U) \subseteq \hat{A}$ is open.

For $U = e^{2\pi i(-1/2, 1/2)}$, $P(A, U) = \{1\}$, so $\{1\}$ is open $\Rightarrow \{x\}$ open

$\forall x \in \hat{A}$. Thus $\hat{A}$ is discrete



Now suppose A is discrete. Idea: Use a compact exhaustion of A to show A is countable then prove $\hat{A}$ sequentially compact. $\square$

Thm [Pontryagin duality] For A an LCA group,

$$\text{eval}: A \longrightarrow \hat{\hat{A}}$$

is an isomorphism of LCA groups.

Functoriality

Write LCA for the category of LCA grps + cts gp homomorphisms.

• composition = composition of functions is associative

• identity ✓

$$\hat{(\cdot)} = LCA(-, S') : LCA \longrightarrow LCA^{\text{op}}$$

$$\begin{array}{ccc} A & & \hat{A} \\ f \downarrow & \longmapsto & f^* \uparrow \\ B & & \hat{B} \end{array} \quad \begin{array}{c} x \cdot f \\ \uparrow \\ x \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow f^*x & \downarrow x \\ & & S' \end{array}$$

satisfies $\text{id}_A^* = \text{id}_{\hat{A}}$ and $(g \circ f)^* = f^* \circ g^*$:

$$(g \circ f)^*(x) = x \circ (g \circ f) = (x \circ g) \circ f = f^*(g^*x)$$

Claim $\hat{(\cdot)}$ is an anti-equivalence of categories.

$F: C \longrightarrow D$ is an equiv of cats when $\exists G: D \longrightarrow C$
 s.t. $\alpha: G \circ F \xRightarrow{\cong} id_C$ and $\beta: F \circ G \xRightarrow{\cong} id_D$.

$$\begin{array}{ccc}
 c & & \\
 f \downarrow & & \\
 c' & &
 \end{array}
 \quad
 \begin{array}{ccc}
 GFc & \xrightarrow[\cong]{\alpha_c} & c \\
 GFf \downarrow & \circlearrowleft & \downarrow f \\
 GFc' & \xrightarrow[\cong]{\alpha_{c'}} & c'
 \end{array}$$

For $(\hat{}): LCA \xrightarrow{\sim} LCA^{\text{op}}$ need $\hat{\hat{}} \cong id_{LCA}$

$$\begin{array}{ccc}
 A & \xrightarrow{\text{eval}} & \hat{A} \\
 f \downarrow & & \downarrow f^{**} \\
 B & \xrightarrow{\text{eval}} & \hat{B}
 \end{array}
 \quad
 \begin{array}{ccc}
 a & \mapsto & \text{eval}_a \\
 \downarrow & & \downarrow \\
 f(a) & \mapsto & \text{eval}_{f(a)} \\
 \hat{B} & \xrightarrow{f^*} & \hat{A}
 \end{array}
 \quad
 \begin{array}{ccc}
 \eta & \mapsto & \eta \cdot f \\
 & & \downarrow \\
 & & \chi(\eta \circ f) \\
 & & = (\eta \circ f)(a) \\
 & & = \eta(f(a)) \\
 & & = \text{eval}_{f(a)}(\eta)
 \end{array}$$

E.g. Consider $A = \mu_{p^\infty} := \{z \in S^1 \mid z^{p^n} = 1 \text{ for some } n \in \mathbb{N}\}$, p prime.

This is the Prüfer ^{P-}group. What topology shall we give it?
 - w/ discrete top!

$$\star \mu_p \hookrightarrow \mu_{p^2} \hookrightarrow \mu_{p^3} \hookrightarrow \dots$$

with union = colimit μ_{p^∞} :

$$\mu_{p^\infty} = \varinjlim_n \mu_{p^n}.$$

Hit $\star$ with $(\hat{})$:

$$\hat{\mu}_p \longleftarrow \hat{\mu}_{p^2} \longleftarrow \hat{\mu}_{p^3} \longleftarrow \dots$$

$$\mathbb{Z}/p\mathbb{Z} \longleftarrow \mathbb{Z}/p^2\mathbb{Z} \longleftarrow \mathbb{Z}/p^3\mathbb{Z} \longleftarrow \dots$$

Fact Representable functors take colimits to limits so

$$\widehat{\mu}_{p^\infty} = \widehat{\operatorname{colim}_n \mu_{p^n}} = \lim_n \widehat{\mu_{p^n}} = \lim_n \mathbb{Z}/p^n \mathbb{Z} = \mathbb{Z}_p$$

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p -adic integers.

Category Theory in Context