Locally compact Abelian groups

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Topology review:

Set X. A topology on X is a collection of subset $\tau \subseteq 2^X$, satisfying:

(0) $\emptyset, X \in \tau$

(1) τ is closed under arbitrary unions

(2) ——— " ——— finite intersections.

E.g. $\mathbb{R}$ with $\tau = \{ U \subseteq \mathbb{R} \mid \forall x \in U \; \exists r > 0 \text{ s.t. } B_r(x) \subseteq U \}$

Defn A subset $K \subseteq X$ of a top'l space is compact when every open cover of K has a finite subcover.

Defn If X, Y top'l spaces, a function $f: X \to Y$ is **continuous** when $\forall U \subseteq Y$ open, $f^{-1}U \subseteq X$ is open.

Exc If topologies of X, Y arise from metrics, then this equivalent to ε-δ continuity:

$$\forall x \in X \ \forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } f B_\delta(x) \subseteq B_\varepsilon(f(x))$$

Defn A *topological group* G is a topological space G equipped with continuous multiplication $\mu : G \times G \to G$, $(g,h) \mapsto gh$ and inversion $\iota : G \to G$, $g \mapsto g^{-1}$ making G a group.

$$\begin{tikzcd} & G \arrow[r, "\Delta"] \arrow[dl] & G\times G \arrow[d, "\iota \times id"] \\ e \arrow[dr] & & G \times G \arrow[dl, "\mu"'] \\ & G & \end{tikzcd}$$

$$\begin{tikzcd} G\times G\times G \arrow[r, "id \times \mu"] \arrow[d, "\mu \times id"'] \arrow[dr, phantom, "\circlearrowright"] & G \times G \arrow[d, "\mu"] \\ G \times G \arrow[r, "\mu"'] & G \end{tikzcd}$$

Note For $g \in G$ and $U \subseteq G$ open with $g \in U$, the set $g^{-1} \cdot U = \{ g^{-1} u \mid u \in U \}$ is an open neighborhood of $e \in G$ homeomorphic to U.

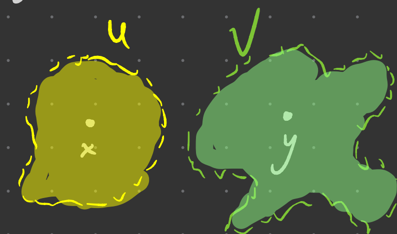
$$\begin{tikzcd} g^{-1}U \arrow[r, bend left, "cts"] & U \arrow[l, bend left, "cts"] \end{tikzcd}$$ composites $=$ id.

Defn A space X is *locally compact* when $\forall x \in X$ $\exists K \subseteq X$ compact, $U \subseteq X$ open with $x \in U \subseteq K$

K

A topological group is locally compact iff e has a compact neighborhood.

<u>Dfn</u> A space X is Hausdorff when $\forall\, x \neq y \in X$ $\exists\, U, V \subseteq X$ open with $x \in U$, $y \in V$, $U \cap V = \emptyset$.



<u>Prop</u> A topological group G is Hausdorff iff $\{e\} \subseteq G$ is closed.

complement of open

<u>Pf</u> ($\Rightarrow$) WTS: $G \setminus \{e\}$ is open.

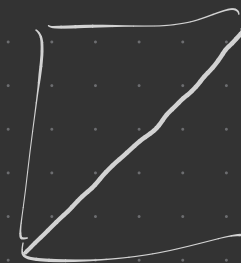
For $g \in G \setminus \{e\}$, by H'ff choose $U_g \ni g$, $V_g \ni e$ open with $U_g \cap V_g = \emptyset$. Then $G \setminus \{e\} = \bigcup_{g \in G} U_g$ open ✓.

($\Leftarrow$) From point-set topology: X is H'ff iff

$$\Delta = \{(x,x) \in X \times X \mid x \in X\} \subseteq X \times X \text{ is closed.}$$

Want: $\Delta = f^{-1}C$, $\quad f : G \times G \xrightarrow{\text{cts}} G$

$$C \subseteq G \text{ closed}$$



Take $f(g,h) = g \cdot h^{-1}$ which is cts

Then $f^{-1}\{e\} = \Delta$ is closed. $\square$

Henceforth, A is a locally compact Hausdorff topological group; shorthand LCA group. (the H is silent)

Defn The Pontryagin dual of A is

$$\hat{A} := \mathrm{Hom}_{\mathrm{cts}}(A, S^1)$$

$$\begin{array}{ccc} \mathbb{C}^\times & = & GL_1\mathbb{C} \\ \vee\mid & & \vee\mid \\ S^1 & = & U_1(\mathbb{C}) \end{array}$$

the group of unitary characters of A under pointwise multiplication.

We endow $\hat{A}$ with the compact open topology: the coarsest topology with opens $P(K, U) := \{\chi \in \hat{A} \mid \chi K \subseteq U\}$ for all $K \subseteq A$ compact, $U \subseteq S^1$ open.

Note Opens of $\hat{A}$ = unions of finite intersections of $P(K_i, U_i)$.

E.g. $\hat{S^1} \cong \mathbb{Z}$ w/ discrete topology

$\hat{\mathbb{R}} \cong \mathbb{R}$ w/ standard topology

$\hat{A} \cong A$ for A finite discrete.

Prop If A is an LCA group, then $\hat{A}$ is an LCA group.

Pf Suffices to show $e \in \hat{A}$ has a compact neighborhood.

Let K be a compact nbhd of e in A,

$U := e^{2\pi i (-1/4, 1/4)}$.

Claim $\overline{P(K,U)}$ is a compact nbhd of $e \in \hat{A}$.

Pf HW.

Why is $\hat{A}$ Hausdorff?