

Fourier analysis on finite Abelian groups

Let A be a finite Abelian group.

E.g. $A = S^1, \mathbb{R}$ but not finite

- $\mathbb{Z}/5\mathbb{Z}$, other cyclic groups $C_n = \langle x \mid x^n \rangle \cong \{ e^{2\pi i k/n} \mid k=0,1,\dots,n-1 \}$
- $(\mathbb{Z} \setminus \{0\}, \cdot)$ is a ^{commutative} monoid but no inverses & not finite
- $(\{\pm 1\}, \cdot)$ is a finite group $\cong C_2$
- $C_2 \times C_2 = K_4$ Klein 4-group

Thm Every finite Abelian group is isomorphic to a product of cyclic groups.

Defn A character of A is a homomorphism $\chi: A \rightarrow \mathbb{C}^\times$.

Prop $\text{im } \chi \leq S^1 = \{z \in \mathbb{C} \mid |z|=1\} = U(1)$ (ii)
($\mathbb{C} \setminus \{0\}, \cdot$)

Pf For each $a \in A$, $\exists n \in \mathbb{Z}_{>0}$ s.t. $a^n = 1$ ↖ id of A
(In fact, $n \mid |A|$)

Thus $\underset{\substack{\uparrow \\ \mathbb{C}}}{1} = \chi(1) = \chi(a^n) = \chi(a)^n \implies \chi(a) = \underset{\substack{\uparrow \\ \in S^1}}{e^{2\pi i k/n}} \text{ for some } k \in \mathbb{Z}$

Rmk In fact $\text{im}(\chi) \leq \mu_\infty = \{z \in \mathbb{C} \mid z^n = 1 \text{ for some } n \in \mathbb{Z}\} \cong \mathbb{Q}/\mathbb{Z}$ □

Write $\hat{A} := \{\text{characters of } A\}$. Equipped with pointwise product

$$\chi\eta(a) = \chi(a)\eta(a)$$

this is the Pontryagin dual of A .

Note $\hat{A}$ is Abelian.

• id of $\hat{A}$ is $1: A \rightarrow \mathbb{C}^\times$
 $a \mapsto 1$

• $\chi^{-1} = \overline{\chi}$: $\overline{\chi}(ab) = \overline{\chi(ab)} = \overline{\chi(a)\chi(b)} = \overline{\chi(a)}\overline{\chi(b)} = \overline{\chi(a)}\overline{\chi(b)} \checkmark$

Q What are the characters of C_n ?

$$\langle x | x^n \rangle$$

A $\chi: C_n \rightarrow \mathbb{C}^\times$ specified by $\chi(x) + \chi(x^k) = \chi(x)^k$

but — as before — $\chi(x) = e^{2\pi i l/n}$ for some $l \in \mathbb{Z}$ or
 $l = 0, 1, \dots, n-1$

Define $\chi_l \in \hat{C}_n$ to be the unique
such character.

$$\text{Then } \chi_l \chi_m(x) = \chi_l(x) \chi_m(x) = e^{2\pi i l/n} e^{2\pi i m/n}$$

$$= \chi_{l+m}(x)$$

in $\mathbb{Z}/n\mathbb{Z}$

$$\text{hence } \hat{C}_n \cong C_n$$

or χ_l for $\gcd(l, n) = 1$ $\chi_1 \mapsto x$

$$z^n = 1 \\ z^k \neq 1, 1 \leq k \leq n-1$$

Thm There is a canonical isomorphism

$$\begin{array}{ccc} A & \xrightarrow{\cong} & \hat{\hat{A}} \\ a \mapsto & & \hat{a} \\ & & \downarrow \text{eval}_a \\ & & \mathbb{C}^x \end{array} \quad \begin{array}{c} \chi \\ \downarrow \\ \chi(a) \end{array}$$

Pf $\text{eval}_{ab}(\chi) = \chi(ab) = \chi(a)\chi(b) = \text{eval}_a(\chi) \text{eval}_b(\chi)$ so eval is a homomorphism.

Now proceed by induction:

(a) true for cyclic groups

(b) true for $A, B \Rightarrow$ true for $A \times B$.

(a) Suppose $A = C_n$. Since $\hat{C}_n \cong C_n$, also $\hat{\hat{C}}_n \cong C_n$.
In particular, $|\hat{\hat{A}}| = |A|$. Thus it suffices to show

$\text{eval} : A \rightarrow \hat{\hat{A}}$ is injective $\Leftrightarrow |\ker(\text{eval})| = 1$.

Suppose $\text{eval}_a = 1 : \hat{A} \rightarrow \mathbb{C}^\times$ So $a \in \ker(\text{eval})$
 $x \mapsto 1 = x(a)$

iff $x(a) = 1 \quad \forall x \in \hat{A}$. Since $A = C_n$, each $x \in \hat{A}$ is of the form $x_k : x \mapsto e^{2\pi i k l / n}$. Write $a = x^k \in C_n$.

Then $1 = x_k(a) = e^{2\pi i k l / n} \Rightarrow n/k \Rightarrow a = 1$.

Hence eval is injective $\Rightarrow$ isomorphism.

(b) Follows from $\widehat{A \times B} \cong \hat{A} \times \hat{B}$ — exc to finish. $\square$

$$\begin{array}{ccc} A \times B & & A \quad B \\ x \downarrow & \xrightarrow{\quad} & x(-, 1) \downarrow \quad \downarrow x(1, -) \\ \mathbb{C}^\times & & \mathbb{C}^\times \quad \mathbb{C}^\times \end{array}$$