

Inversion FormulaDefn For $\lambda > 0$, the Gauss kernel is

$$h_\lambda: \mathbb{R} \longrightarrow \mathbb{R}$$

$$x \longmapsto \int_{\mathbb{R}} e^{-\lambda|t|} e^{2\pi i t x} dt.$$

Lemma 1 In fact, $h_\lambda(x) = \frac{2\lambda}{4\pi^2 x^2 + \lambda^2}$ and $\int_{\mathbb{R}} h_\lambda(x) dx = 1$.It follows that for $\lambda > 0$, $h_\lambda(x) = \frac{1}{\lambda} h_1\left(\frac{x}{\lambda}\right)$.

Pf We compute

$$h_{\lambda}(x) = \int_0^{\infty} e^{2\pi i t x - \lambda t} dt + \int_{-\infty}^0 e^{2\pi i t x + \lambda t} dt$$

$$= \left. \frac{e^{2\pi i t x - \lambda t}}{2\pi i x - \lambda} \right|_0^{\infty} + \left. \frac{e^{2\pi i t x + \lambda t}}{2\pi i x + \lambda} \right|_{-\infty}^0$$

$$= \frac{1}{\lambda - 2\pi i x} + \frac{1}{\lambda + 2\pi i x}$$

$$= \frac{2\lambda}{\lambda^2 + 4\pi^2 x^2}.$$

Further,

$$\int_{\mathbb{R}} h_{\lambda}(x) dx = \frac{2}{\lambda} \int_{\mathbb{R}} \frac{1}{1 + \left(\frac{2\pi x}{\lambda}\right)^2} dx$$

$$u = \frac{2\pi x}{\lambda} \quad du = \frac{2\pi}{\lambda} dx$$

$$= \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{1+u^2} du$$

$$= \frac{1}{\pi} \arctan(u) \Big|_{-\infty}^{\infty}$$

$$= 1. \quad \square$$

Lemma 2 If $f \in L^1_{bc}(\mathbb{R})$, then for all $\lambda > 0$,

$$f * h_\lambda(x) = \int_{\mathbb{R}} e^{-\lambda|t|} \hat{f}(t) e^{2\pi i x t} dt$$

Pf Let's compute:

$$f * h_\lambda(x) = \int_{\mathbb{R}} f(x-y) h_\lambda(y) dy$$

$$= \int_{\mathbb{R}} f(x-y) \int_{\mathbb{R}} e^{-\lambda|t|} e^{2\pi i t y} dt dy$$

$$= \int_{\mathbb{R}} e^{-\lambda|t|} \int_{\mathbb{R}} f(x-y) e^{2\pi i t y} dy dt \quad y \leftarrow x-y$$

$$= \int_{\mathbb{R}} e^{-\lambda|t|} \int_{\mathbb{R}} f(y) e^{2\pi i t(x-y)} dy dt$$

$$= \int_{\mathbb{R}} e^{-\lambda|t|} e^{2\pi i x t} \int_{\mathbb{R}} f(y) e^{-2\pi i t y} dy dt$$

$$= \int_{\mathbb{R}} e^{-\lambda|t|} e^{2\pi i x t} \hat{f}(t) dt. \quad \square$$

Lemma 3 For all $f \in L^1_{loc}(\mathbb{R})$, $x \in \mathbb{R}$,

$$\lim_{\lambda \rightarrow 0^+} f * h_\lambda(x) = f(x).$$

Pf Again, calculate:

$$f * h_\lambda(x) - f(x) = \int_{\mathbb{R}} f(x-y) h_\lambda(y) dy - f(x)$$

$\Rightarrow f(x) = \int_{\mathbb{R}} f(x) h_\lambda(y) dy$
 $\uparrow$

$$= \int_{\mathbb{R}} (f(x-y) - f(x)) h_\lambda(y) dy \quad \left[\int_{\mathbb{R}} h_\lambda = 1 \right]$$

$$= \int_{\mathbb{R}} (f(x-y) - f(x)) \frac{1}{\lambda} h_1(y/\lambda) dy$$

$$u = \frac{y}{\lambda}, \quad du = \frac{1}{\lambda} dy$$

$$= \int_{\mathbb{R}} (f(x-\lambda u) - f(x)) h_1(u) du$$

Since $f \in L^1(\mathbb{R})$, $\exists C > 0$ st. $|f(x)| \leq C \quad \forall x \in \mathbb{R}$.

Thus the integrand is dominated by $2Ch_1(u)$.

As $\lambda \rightarrow 0^+$, $f(x - \lambda u) \rightarrow f(x)$ locally uniformly in u .

So by dominated convergence,

$$f * h_\lambda(x) - f(x) \xrightarrow{\lambda \rightarrow 0^+} 0 \quad \square$$

Then [inversion formula] Let $f \in L^1_{bc}(\mathbb{R})$ and assume $\hat{f} \in L^1_{bc}(\mathbb{R})$ as well. Then $\forall x \in \mathbb{R}$,

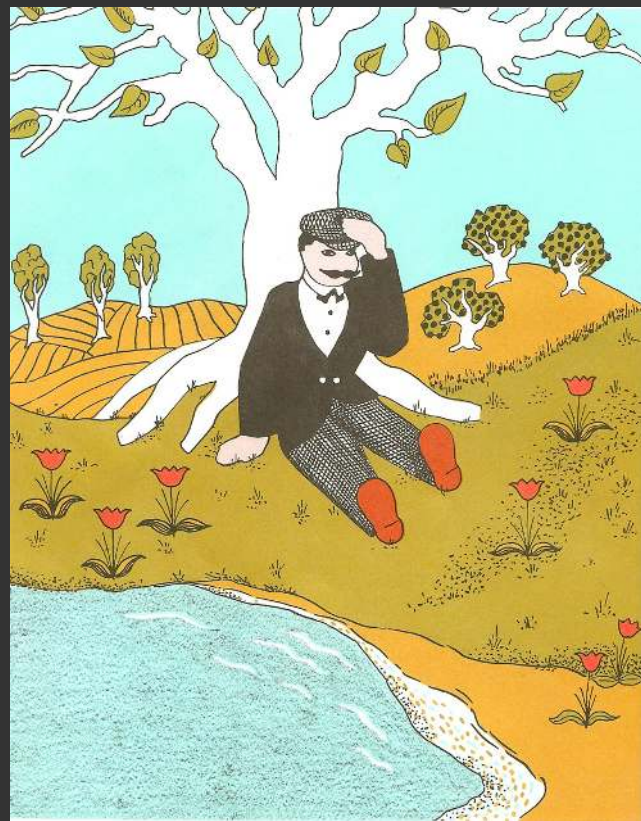
$$\hat{\hat{f}}(x) = f(-x)$$

$$\text{Cor } \hat{\hat{\hat{f}}} = f$$

Cor



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Pf The claim is equivalent to

$$f(x) = \int_{\mathbb{R}} \hat{f}(y) e^{2\pi i x y} dy.$$

For $\lambda > 0$,

$$f * h_{\lambda}(x) = \int_{\mathbb{R}} e^{-\lambda|t|} \hat{f}(t) e^{2\pi i x t} dt$$

by Lemma 2. As $\lambda \rightarrow 0^+$, the LHS $\rightarrow f(x)$ by Lemma 3.

The RHS integrand is dominated by $|\hat{f}(t)|$, so by dominated convergence,

$$f(x) = \lim_{\lambda \rightarrow 0^+} \int_{\mathbb{R}} e^{-\lambda|t|} \hat{f}(t) e^{2\pi i x t} dt = \int_{\mathbb{R}} \hat{f}(t) e^{2\pi i x t} dt. \quad \square$$

Cor $\hat{I} : \mathcal{S} \xrightarrow{\cong} \mathcal{S}$ with characteristic polynomial $X^4 - 1$.
 $\Rightarrow$ the only eigenvalues of $\hat{I}$ are $\pm 1, \pm i$.

Prop Let $f(x) = e^{-\pi x^2}$. Then $f \in \mathcal{S}$ and $\hat{f} = f$.

Pf Observe $f'(x) = -2\pi x e^{-\pi x^2} = -2\pi x f(x)$.

In fact (exc), f is the unique sol'n to this diff'l eqn up to scalar multiplication. We have $f \in \mathcal{S}$ (why?) so $\hat{f} \in \mathcal{S}$ and we can compute

$$\begin{aligned} (\hat{f})'(y) &= \int_{\mathbb{R}} (-2\pi i x) e^{-\pi x^2} e^{-2\pi i x y} dx \\ &= i \int_{\mathbb{R}} (e^{-\pi x^2})' e^{-2\pi i x y} dx \end{aligned}$$

$$u = e^{-2\pi ixy} \quad dv = (e^{-\pi x^2})' dx$$

$$du = -2\pi i y e^{-2\pi ixy} dx \quad v = e^{-\pi x^2}$$

$$= iuv \Big|_{-\infty}^{\infty} - 2\pi y \int_{\mathbb{R}} f(x) e^{-2\pi ixy} dx$$

$$= -2\pi y \hat{f}(y).$$

By the diff' l eqn, $\hat{f}(y) = ce^{-\pi^2 y}$ for some constant c .

Since $\hat{\hat{f}}(x) = f(-x)$, know $c^2 = 1 \Rightarrow c = \pm 1$. Since

$\hat{f}(0) = \int_{\mathbb{R}} e^{-\pi x^2} dx > 0$, we must have $c = 1$. $\square$

Cor $\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$

 No elementary expression
for anti-derivative of
 e^{-x^2}

Pf Let $f(x) = e^{-\pi x^2}$. By the proposition,

$$\begin{aligned} 1 = f(0) &= \hat{f}(0) = \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi i x \cdot 0} dx \\ &= \int_{\mathbb{R}} e^{-\pi x^2} dx. \end{aligned}$$

$$\text{Now } \int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi} \int_{\mathbb{R}} e^{-\pi u^2} du = \sqrt{\pi}. \quad \square$$

$$u = \frac{x}{\sqrt{\pi}}$$

$$du = \frac{1}{\sqrt{\pi}} dx$$