

Defn For $f \in L^1_{bc}(\mathbb{R})$, the Fourier transform of f is

$$\begin{aligned} \hat{f} : \mathbb{R} &\longrightarrow \mathbb{C} \\ \gamma &\longmapsto \hat{f}(\gamma) := \int_{\mathbb{R}} f(x) e^{-2\pi i \gamma x} dx. \end{aligned}$$

Thm Let $f \in L^1_{bc}(\mathbb{R})$.

- (a) If $g(x) = f(x)e^{2\pi i a x}$, then $\hat{g}(\gamma) = \hat{f}(\gamma - a)$ ✓
- (b) If $g(x) = f(x - a)$, then $\hat{g}(\gamma) = \hat{f}(\gamma)e^{-2\pi i a \gamma}$ ✓
- (c) If $g \in L^1_{bc}(\mathbb{R})$ and $h = f * g$, then $\hat{h}(\gamma) = \hat{f}(\gamma)\hat{g}(\gamma)$
- (d) If $g(x) = f\left(\frac{x}{\lambda}\right)$ for $\lambda > 0$, then $\hat{g}(\gamma) = \lambda \hat{f}(\lambda \gamma)$ ✓

(e) If $g(x) = -2\pi i x f(x)$, $g \in L^1_{bc}(\mathbb{R})$, then $\hat{f} \in C^1(\mathbb{R})$ with $\hat{f}'(\gamma) = \hat{g}(\gamma)$.

(f) Suppose $f \in C^1(\mathbb{R})$ and $f, f' \in L^1_{bc}(\mathbb{R})$. Then $\hat{f}'(\gamma) = 2\pi i \gamma \hat{f}(\gamma)$,
in particular, $\gamma \mapsto \gamma \hat{f}(\gamma)$ is bounded. $|\gamma \hat{f}(\gamma)| \leq C$ for some $C > 0$
 $\Rightarrow |\hat{f}(\gamma)| \leq \frac{C}{|\gamma|} \rightarrow 0$ as $\gamma \rightarrow \pm\infty$.

(g) Suppose $f \in C^2(\mathbb{R})$ and $f, f', f'' \in L^1_{bc}(\mathbb{R})$. Then $\hat{f} \in L^1_{bc}(\mathbb{R})$.

Pf (c) We compute

$$\hat{h}(\gamma) = \int_{\mathbb{R}} h(x) e^{-2\pi i \gamma x} dx \quad \text{defn}$$

$$= \int_{\mathbb{R}} \underbrace{\int_{\mathbb{R}} f(x-t) g(t) dt}_{f * g(x)} e^{-2\pi i \gamma x} dx$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-t) e^{-2\pi i \gamma x} dx g(t) dt$$

$$u = x - t \Rightarrow x = u + t, du = dx \\ e^{-2\pi i \gamma x} = e^{-2\pi i \gamma u} e^{-2\pi i \gamma t}$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} f(u) e^{-2\pi i \gamma u} du g(t) e^{-2\pi i \gamma t} dt$$

$$= \hat{f}(\gamma) \hat{g}(\gamma) \quad \hat{f}(\gamma) \quad \square$$

Pf (v) Observe that $\frac{\hat{f}(\gamma) - \hat{f}(t)}{\gamma - t} = \int_{\mathbb{R}} f(x) e^{-2\pi i t x} \frac{e^{-2\pi i \gamma x} - 1}{\gamma - t} dx$

$$u = \gamma - t \rightarrow 0 \Leftrightarrow t \rightarrow \gamma$$

let $\varphi(x, u) = \frac{e^{-2\pi i u x} - 1}{u}$. Then $|\varphi(x, u)| \leq 2\pi |x|$ for $u \neq 0$ exc

and $\varphi(x, u) \rightarrow -2\pi i x$ as $u \rightarrow 0$. The convergence is locally

$$\frac{\partial}{\partial u} \varphi(x, u) \Big|_{u=0}$$

uniform in x . The claim now follows from dominated convergence. $\square$

Pf (f) Since $f \in L^1(\mathbb{R})$, $\exists (s_n), (t_n) \rightarrow \infty$ with $f(s_n), f(t_n) \rightarrow 0$.

$$\begin{aligned} \text{Then } \hat{f}'(\gamma) &= \lim_{n \rightarrow \infty} \int_{-s_n}^{t_n} f'(x) e^{-2\pi i \gamma x} dx \\ &= \lim_{n \rightarrow \infty} \left(\underbrace{f(x) e^{-2\pi i \gamma x}}_{\substack{u = e^{-2\pi i \gamma x} \\ du = -2\pi i \gamma e^{-2\pi i \gamma x}}} \right) \bigg|_{-s_n}^{t_n} + 2\pi i \gamma \int_{-s_n}^{t_n} f(x) e^{-2\pi i \gamma x} dx \quad \substack{dv = f'(x) dx \\ v = f(x)} \\ &= 2\pi i \gamma \hat{f}(\gamma). \quad \square \end{aligned}$$

Pf (g) Applying (f) twice shows $\gamma^2 \hat{f}(\gamma)$ is bounded, so

$|\hat{f}(x)| \leq \frac{C}{|x|^2}$ for some $C \geq 0$. Since $\hat{f}$ is cts its integrable

so $\int_{\mathbb{R}} |\hat{f}| \leq \int_{\mathbb{R}} \frac{C}{|x|^2} dx < \infty$ by comparison test. $\square$

Schwartz space

Let $\mathcal{S} = \mathcal{S}(\mathbb{R}) := \{f \in C^\infty(\mathbb{R}) \mid \sigma_{m,n}(f) < \infty \ \forall m, n \in \mathbb{N}\}$ where

$$\sigma_{m,n}(f) := \sup_{x \in \mathbb{R}} |x^m f^{(n)}(x)| < \infty$$

Q What do Schwartz functions "look like"?

A f (and all $f^{(n)}$) go to 0 at $\pm\infty$ faster than $\frac{1}{\text{polynomial}}$. E.g. $f(x) = e^{-x^2}$.

Prop We have $\mathcal{S} \subseteq L'_{bc}(\mathbb{R})$, and for $f \in \mathcal{S}$, $\hat{f} \in \mathcal{S}$ as well.

Pf Let $f \in \mathcal{S}$, so f is bdd cts and $|(1+x^2)f(x)| \leq C$ for some $C > 0$, so $\int_{\mathbb{R}} |f| \leq C \int_{\mathbb{R}} \frac{1}{1+x^2} dx = C\pi < \infty$.

Thus $f \in L'_{bc}(\mathbb{R})$. ($\int \frac{1}{1+x^2} dx = \arctan(x) + C'$)

By (a), $\hat{f} \in C^\infty(\mathbb{R})$ with $\hat{f}^{(n)}(x) = \overbrace{(-2\pi i x)^n f(x)}$

By (f), $\widehat{f^{(n)}}(\gamma) = (2\pi i \gamma)^n \hat{f}(\gamma)$.

Thus each function $\gamma^n \hat{f}^{(n)}(\gamma)$ is a Fourier transform of some function in $\mathcal{S}$ and hence is bounded. □

Up next inversion theorem for Fourier transforms:

$$\hat{\hat{f}}(x) = f(-x) \quad \text{for } f \in \mathcal{S}$$

Proof will use Gauss kernels:

for $\lambda > 0$, $x \in \mathbb{R}$

$$h_\lambda(x) := \int_{\mathbb{R}} e^{-\lambda|t|} e^{2\pi i t x} dt.$$