

Eigenbasis Method

L operator on $L^2(S')$ or $L^2([a, b])$ (x coord)

T operator on $L^2(\mathbb{R}_{>0})$ or $L^2(\mathbb{R}_{>0})$ (t coord)

Goal Find $u: S' \times \mathbb{R}_{>0} \rightarrow \mathbb{C}$ satisfying $L(u) = T(u)$
and boundary, initial conditions.

Separation of variables seeks a solution of the form

$$u(x, t) = \sum \phi_n(x) \psi_n(t)$$

eigenvalues
p. real

(1) Boundary conditions on x determine $\mathcal{D}(L)$. Show L Hermitian.

(2) Find an eigenbasis for L : orthogonal basis (ϕ_n) of $\mathcal{H} = L^2(S')$
with $\phi_n \in \mathcal{D}(L)$, $L\phi_n = \lambda_n \phi_n$ for some $\lambda_n \in \mathbb{R}$.

(3) For each n solve the ODE $T \dot{\psi}_n = \lambda_n \psi_n$ for ψ_n

(4) Know L is diagonalizable wrt (ϕ_n) ; hope T is as well wrt ψ_n .

Then $u(x,t) = \sum \phi_n(x) \psi_n(t)$ is a formal solution to

$Lu = Tu$. Remains to check initial condition, convergence,

and validity of $T\Sigma(\dots) = \Sigma T(\dots)$ in (4).

$A \in F^{n \times n}$ diag'le

then $\exists$ basis $v_1, \dots, v_n$ of F^n s.t.

(1) $UAU^{-1} = \text{diag}(\lambda_1, \dots, \lambda_n) \Leftrightarrow$ (2) $Av_i = \lambda_i v_i$

Tools for uniform convergence

Recall a sequence of fns $f_n: X \subseteq \mathbb{C} \rightarrow \mathbb{C}$ converges uniformly to $f: X \rightarrow \mathbb{C}$ when $\forall \epsilon > 0 \exists N$ s.t. $\forall z \in X, n > N, |f_n(z) - f(z)| < \epsilon$.

$\uparrow$ ind of z !

Lemma $(f_n: X \rightarrow \mathbb{C})$ converges uniformly to $f: X \rightarrow \mathbb{C}$ iff

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{\infty} = 0 \quad \text{I.e. uniform convergence} = L^{\infty} \text{ convergence}$$

$$\text{for } \|g\|_{\infty} = \sup_{z \in X} |g(z)|$$

Pf Let $d_n := \|f_n - f\|_{\infty} = \sup_{z \in X} |f_n(z) - f(z)|$.

$(\Leftarrow)$ Suppose $\lim_{n \rightarrow \infty} d_n = 0$. Since $|f_n(z) - f(z)| \leq d_n \quad \forall z \in X$,

$f_n(z) \xrightarrow[n \rightarrow \infty]{} f(z)$ with rate independent of z ✓

$(\Rightarrow)$ Suppose $\forall \varepsilon > 0 \exists N$ s.t. $\forall z \in X, n > N$, we have $|f_n(z) - f(z)| < \frac{\varepsilon}{2}$.

Then $d_n \leq \frac{\varepsilon}{2} < \varepsilon$ so $d_n \rightarrow 0$. □

Weierstrass M-test Suppose $\emptyset \neq X \subseteq \mathbb{C}$, $g_n: X \rightarrow \mathbb{C}$,

$M_n > 0$ with $\sum M_n$ convergent, and $|g_n(z)| \leq M_n \quad \forall z \in X$.

Then $\sum g_n(z)$ converges absolutely and uniformly to some $f: X \rightarrow \mathbb{C}$.

Pf Observe $\left| \sum_{n=k}^m g_n(z) \right| \leq \sum_{n=k}^m |g_n(z)| \leq \sum_{n=k}^m M_n = \left| \sum_{n=k}^m M_n \right|$

Since $\sum M_n$ converges, its partial sums satisfy the Cauchy criterion, so the same holds for $\sum g_n(z)$, ind. of z .

I.e. $\sum g_n$ is uniformly Cauchy $\Rightarrow$ uniformly convergent.

All this holds for $\sum |g_n|$ as well, so $\sum g_n$ is absolutely conv. too. $\square$

Limits vs. integrals and derivatives

Thm Let $(f_n: [a, b] \rightarrow \mathbb{C})$ be a sequence of ^{Riemann} integrable fns
converging uniformly to $f: [a, b] \rightarrow \mathbb{C}$. Then

$$\int_a^b f(x) dx = \int_a^b \left(\lim_{n \rightarrow \infty} f_n(x) \right) dx = \lim_{n \rightarrow \infty} \int_a^b f_n(x) dx.$$

Pf AW. $\square$

Limits must converge with derivatives too! Right? ... right?

E.g. Set $f_n(x) = |x|^{1+1/n} \xrightarrow{n \rightarrow \infty} |x|$ uniformly in x

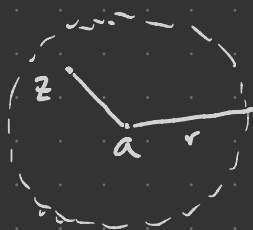
We have $f'_n(0) = 0$, but $|x|$ is not diff'ble at 0.

Nonetheless,

Thm Let $\emptyset \neq X \subseteq \mathbb{C}$ open, $f_n: X \rightarrow \mathbb{C}$ diff'ble converging pointwise to $f: X \rightarrow \mathbb{C}$. Suppose each f'_n is cts and the sequence f'_n converges uniformly to some $g: X \rightarrow \mathbb{C}$. Then f is diff'ble and $f'(z) = g(z)$.

Cor If $\sum g'_n(z)$ converges uniformly, each g'_n is cts, and $\sum g_n(z)$ converges, then $\frac{d}{dz} \left(\sum g_n(z) \right) = \sum g'_n(z)$. Pf Ex. $\square$

Pf Thm Fix $a \in X$. WLOG, assume $X = B_r(a)$ for some $r > 0$.



For fixed $z \in B_r(a)$, define $u_z: [0, 1] \rightarrow \mathbb{C}$
 $t \mapsto tz + (1-t)a$

Then $u'_z(t) = z - a$ and $|u_z(t) - a| \leq |z - a|$,
with $u_z(0) = a$, $u_z(1) = z$.

Compute $I = \lim_{n \rightarrow \infty} \int_0^1 f'_n(u_z(t)) u'_z(t) dt$ in two ways.

By substitution, $I = \lim_{n \rightarrow \infty} (f_n(u_z(1)) - f_n(u_z(0)))$
 $= f(z) - f(a)$. $\odot$

On the other hand, $f'_n(u_z(t))$ converges unif on $[0, 1]$,

$$\text{so } I = \int_0^1 \left(\lim_{n \rightarrow \infty} f'_n(u_z(t)) u'_z(t) \right) dt$$

$$= \int_0^1 g(u_z(t)) (z-a) dt$$

$$= (z-a) \int_0^1 g(u_z(t)) dt \quad \textcircled{2}$$

$$\int_0^1 g(a) dt$$

Since $\textcircled{1} = \textcircled{2}$, $\frac{f(z) - f(a)}{z-a} - g(a) = \left(\int_0^1 g(u_z(t)) dt \right) - g(a)$

$$= \int_0^1 (g(u_z(t)) - g(a)) dt$$

Thus $\left| \frac{f(z) - f(a)}{z-a} - g(a) \right| \leq \int_0^1 |g(u_z(t)) - g(a)| dt \xrightarrow{z \rightarrow a} 0$

$$\text{so } f'(z) = \lim_{z \rightarrow a} \frac{f(z) - f(a)}{z - a} = g(a). \quad \square$$

Back to PDEs ...

Wave equation on S' Given $f, g \in L^2(S')$, find $u: S' \times \mathbb{R}_{>0} \rightarrow \mathbb{C}$

$$\text{s.t. (D) } u(-, t) \in C^2(S'), \quad u(x_0, -) \in C^2(\mathbb{R}_{>0})$$

$$\text{(IV) } \lim_{t \rightarrow 0^+} u(x, t) = f(x), \quad \lim_{t \rightarrow 0^+} \frac{\partial u}{\partial t}(x, t) = g(x).$$

$$\text{(PDE) } -\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial t^2}.$$

$$\text{Apply the eigenbasis method: } L = -\frac{\partial^2}{\partial x^2}, \quad T = -\frac{\partial^2}{\partial t^2}$$

Know L is Hermitian with eigenbasis $(e_n)_{n \in \mathbb{Z}}$ and associated eigenvalues $\lambda_n = 4\pi^2 n^2 = (2\pi n)^2 > 0$. Set $\kappa_n = |2\pi n|$.

Have $f = \sum \hat{f}(n) e_n$, $g = \sum \hat{g}(n) e_n$ in L^2

Our ODE is $T\psi = \lambda_n \psi$, i.e. $\psi'' = -\kappa_n^2 \psi$ with solutions

$$\psi_n(t) = C_0 \cos(\kappa_n t) + \frac{C_1}{\kappa_n} \sin(\kappa_n t) \quad \text{for } n \neq 0.$$

where $C_0 = \psi_n(0) = \hat{f}(n)$, $C_1 = \psi'_n(0) = \hat{g}(n)$

$$\text{i.e.} \quad \psi_n(t) = \hat{f}(n) \cos(\kappa_n t) + \frac{\hat{g}(n)}{\kappa_n} \sin(\kappa_n t)$$

Thus the wave equation has formal sol'n

$$u(x,t) = \hat{f}(0) + t\hat{g}(0) + \sum_{n \neq 0} e_n(x) \psi_n(t)$$

See 11.3 Hsu for proving this converges, etc.