

Partial differential equations

Heat

- Wire with uniform density ρ .
- $u(x, t)$ = temp of wire at pos'n x time t

Q Given initial condition $u(x, 0) = f(x)$,
what is $u(x, t)$?

A Will be specified by a differential equation
(+ boundary conditions)

Defn $q(x, t)$ = rate of flow of heat energy per unit length at (x, t)

Fourier's law of heat conduction

$$q = -k \frac{\partial u}{\partial x}$$

↑
thermal conductivity

Defn $Q(x,t)$ = heat density, so

$$\frac{\partial Q}{\partial t} = c_p \frac{\partial u}{\partial t}$$

↑
specific heat capacity of medium

By conservation of energy,

$$\frac{\partial Q}{\partial t} = - \frac{\partial q}{\partial x}$$

$$\therefore \frac{\partial u}{\partial t} = \frac{-1}{c_p} \frac{\partial q}{\partial x} = \frac{k}{c_p} \frac{\partial^2 u}{\partial x^2}$$

This gives us the heat equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

(in units with $\frac{k}{c\rho} = 1$)

Full specification for heat eq'n on circle:

Given $f \in L^2(S^1)$, find $u: S^1 \times \mathbb{R}_{>0} \rightarrow \mathbb{R}$ such that

(D) For fixed $t_0 > 0$, $u(-, t_0) \in C^2(S^1)$, and for fixed $x_0 \in S^1$, $u(x_0, -) \in C^1(\mathbb{R}_{>0})$

(IV) $\forall x \in S^1$, $\lim_{t \rightarrow 0^+} u(x, t) = f(x)$

(PDE) $-\frac{\partial^2 u}{\partial x^2} = -\frac{\partial u}{\partial t}$

Note Discrete
initial condition
is OK!

Note (PDE) is equiv to $\Delta u = -\frac{\partial u}{\partial t}$ ($\Delta = \nabla^2$) notation
 $\uparrow$
Laplacian (positive operator)

★ LAND OF ★
WISHFUL
THINKING

Recall $(e_n)_{n \in \mathbb{Z}}$ is an eigenbasis for Δ

Since $f \in L^2(S')$, $f(x) = \sum \hat{f}(n) e_n(x)$ in L^2 .

Let $\psi_n(t) = \widehat{u(-, t)}(n)$. Since $u(-, t) \in C^2(S') \subseteq L^2(S')$

have $u(x, t) = \sum \psi_n(t) e_n(x)$ in L^2 .

Know Δ diagonalizes $\nearrow$ Hope: $-\frac{\partial}{\partial t}$ diagonalizes it as well.

If so,

$$\Delta(u)(x,t) = \sum 4\pi^2 n^2 e_n(x) \psi_n(t)$$

$$\frac{-\partial u}{\partial t}(x,t) = \sum (-1) e_n(x) \psi'_n(t) \quad \text{②} \quad \frac{\partial}{\partial t} \text{ commutes with } \sum_{n \in \mathbb{Z}}$$

Thus we seek functions $\psi_n(t)$ cts at 0 such that

$$-\psi'_n(t) = 4\pi^2 n^2 \psi_n(t) \quad \text{and} \quad \psi_n(0) = \hat{f}(n).$$

$$\frac{d}{dt} e^{at} = a e^{at}$$

$$\text{②} \quad \lim_{t \rightarrow 0^+} \text{ commutes with } \sum_{n \in \mathbb{Z}} ?$$

Q Solution for $\psi_n(t)$?

A $\psi_n(t) = \hat{f}(n) e^{-4\pi^2 n^2 t}$

Thus

$$u(x,t) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e_n(x) e^{-4\pi^2 n^2 t}$$

is our candidate solution.

Now need to formally justify:

regularity: $\sum \hat{f}(n) e_n(x) e^{-4\pi^2 n^2 t}$ converges to C^∞ fn of x , C^∞ fn of t .

$$\text{IV} \quad \lim_{t \rightarrow 0^+} \|u(x,t) - f(x)\|_{L^2} = 0.$$

↑ or L^∞ for $f \in C^1(S^1)$

uniqueness our sol'n is the only one satisfying
(D), (IV), (PDE).