

Operators on Hilbert spaces

In preparation for our treatment of PDE's via the eigenbasis method, we develop the language of (Hermitian) operators on a Hilbert space.

Defn Let $\mathcal{H}$ be a Hilbert space. An operator on $\mathcal{H}$ is a linear map $T: \mathcal{D}(T) \rightarrow \mathcal{H}$ where $\mathcal{D}(T) \subseteq \mathcal{H}$.

E.g. $X = \mathbb{S}^1$ or $[a, b]$, $\mathcal{H} = L^2(X)$. Then $D: C^1(X) \rightarrow \mathcal{H}$
 $f \mapsto -if'$
 is an operator on $\mathcal{H}$.

Note $D(f) = \sum_{n \in \mathbb{Z}} 2\pi n \hat{f}(n) e_n$

E.g. For $\mathcal{H}$ a separable Hilbert space with orthonormal basis $\mathcal{B} = (e_n)_{n \geq 1}$, define

$$\mathcal{H}_0 := \left\{ \sum_{n=1}^{\infty} c_n e_n \mid \text{all but finitely many } c_n = 0 \right\}.$$

Define operators μ, ι on $\mathcal{H}$ by formulae

$$\begin{aligned} \mu \left(\sum_{n=1}^{\infty} c_n e_n \right) &= \sum_{n=1}^{\infty} n c_n e_n \\ \iota \left(\sum_{n=1}^{\infty} c_n e_n \right) &= \sum_{n=1}^{\infty} \left(\frac{c_n}{n} \right) e_n \end{aligned}$$

with $\mathcal{D}(\mu) = \mathcal{H}_0$, $\mathcal{D}(\iota) = \mathcal{H}$.

E.g. More generally, for $\alpha: \mathbb{Z}_{\geq 1} \rightarrow \mathbb{C}$ any function,

$$\alpha\left(\sum_{n=1}^{\infty} c_n e_n\right) = \sum_{n=1}^{\infty} \alpha(n) c_n e_n$$

defines a linear operator on $\mathcal{H}$ with $\mathcal{D}(\alpha) = \mathcal{H}_0$.

Defn Such an operator is called a diagonal operator with respect to (e_n) .

E.g. The operator $D: f \mapsto -if'$ on $L^2(S^1)$ is diagonalized by the standard basis $(e_n)_{n \in \mathbb{Z}}$.

Defn An operator T on $\mathcal{H}$ is bounded when $\exists M > 0$ s.t.
 $\forall f \in \mathcal{H}, \|T(f)\| \leq M \|f\|$.

Thm TFAE: (UC) T is uniformly continuous
(CO) T is continuous at $0 \in \mathcal{D}(T)$
(B) T is bounded.

Pf HW. $\square$

Hermitian and positive operators

Defn An operator T on $\mathcal{H}$ is Hermitian (or self-adjoint)
when $\forall f, g \in \mathcal{D}(T), \langle T(f), g \rangle = \langle f, T(g) \rangle$.

Remark Bounded operators on a Hilbert space always admit an adjoint T^* satisfying $\langle T(f), g \rangle = \langle f, T^*g \rangle$.
(This is a consequence of the 'Riesz representation theorem'.)
Hermitian operators satisfy $T = T^*$.

operator : Hermitian operator :: complex #s : real #s

E.g. Consider $D: f \mapsto -if'$ on $L^2(S')$ again, with $D(D) = C'(S')$.

For $f, g \in C'(S')$,

$$\langle D(f), g \rangle = \int_0^1 \underbrace{(-i)f'(x)}_{dv} \underbrace{\overline{g(x)}}_u \underline{dx}$$

$$= (-i)f(x) \overline{g(x)} \Big|_0^1 - \int_0^1 (-i)f(x) \overline{g'(x)} dx$$

$$\begin{aligned}
 &= (-i)(\underbrace{f(1)g(1)}_{\text{yellow checkmark}} - \overset{\text{yellow arrow to 0}}{f(0)g(0)}) + \int_0^1 i f(x) \overline{g'(x)} dx \\
 &= \int_0^1 f(x) \overline{(-i)g'(x)} dx \\
 &= \langle f, D(g) \rangle.
 \end{aligned}$$

Thus D is Hermitian.

Q For which $\alpha: \mathbb{N} \rightarrow \mathbb{C}$ is $\alpha(\sum c_n e_n) = \sum \alpha(n) c_n e_n$ Hermitian? (Hilbert space with orthonormal basis (e_n) .)

$$\begin{aligned}
 \text{A } \langle \alpha(\sum c_n e_n), \sum d_n e_n \rangle &= \langle \sum \alpha(n) c_n e_n, \sum d_n e_n \rangle \quad \forall n. \\
 &= \sum \alpha(n) c_n \overline{d_n} \stackrel{?}{=} \sum c_n \overline{\alpha(n) d_n} \quad \text{iff } \alpha(n) \in \mathbb{R}
 \end{aligned}$$

E.g. The shift $\sigma: \sum c_n e_n \mapsto \sum c_n e_{n+1}$ is a non-Hermitian operator. (moral ex - not Hermitian)

Thm Let T be a Hermitian operator. Then $\forall f \in \mathcal{D}(T)$, $\langle T(f), f \rangle$ is a real number.

Pf By conjugate symmetry, $\langle f, T(f) \rangle = \overline{\langle T(f), f \rangle}$, but the LHS also equals $\langle T(f), f \rangle$ by self-adjointness. Thus $\langle T(f), f \rangle = \overline{\langle T(f), f \rangle}$ is real. $\square$

Defn A Hermitian operator is positive when $\langle T(f), f \rangle \geq 0$ for all f .
 Can have $\langle T(f), f \rangle = 0$ for $f \neq 0$.

Eg. On $L^2(S^1)$, $\Delta: f \mapsto -f''$ with $\mathcal{D}(\Delta) = C^2(S^1)$ is called the Laplacian operator. Observe:

$$\begin{aligned}\langle \Delta(f), g \rangle &= - \int_0^1 \underbrace{f''(x)}_{\text{du}} \underbrace{\overline{g(x)}}_{\text{u}} \underbrace{dx}_{\text{du}} \\ &= - \underbrace{f'(x) \overline{g(x)}}_{\text{du}} \Big|_0^1 + \int_0^1 f'(x) \overline{g'(x)} dx \\ &= \langle f', g' \rangle\end{aligned}$$

Similarly, $\langle f, \Delta(g) \rangle = \langle f', g' \rangle$ so Δ is Hermitian.

Moreover, $\langle \Delta(f), f \rangle = \|f'\|^2 \geq 0$ so Δ is positive.

Eg. A diagonal operator associated with $\alpha: \mathbb{N} \rightarrow \mathbb{C}$ is positive iff $\alpha(n) \geq 0 \forall n$.

Thm A Hermitian operator's eigenvalues are all real. If the operator is positive, then the eigenvalues are also nonnegative.

Pf Suppose $T: \mathcal{D}(T) \rightarrow \mathcal{H}$ is Hermitian with $T(f) = \lambda f$ for some $f \in \mathcal{D}(T) \setminus \{0\}$. Then $\mathbb{R} \ni \langle T(f), f \rangle = \langle \lambda f, f \rangle = \lambda \|f\|^2 \Rightarrow \lambda \in \mathbb{R}$. If T is positive, then $\langle T(f), f \rangle \geq 0$ so $\lambda \geq 0$. $\square$

Thm Let T be a Hermitian operator, $\{u_1, \dots, u_n\}$ a set of eigenvectors of T with distinct eigenvalues $\lambda_1, \dots, \lambda_n$. Then $\{u_1, \dots, u_n\}$ is an orthogonal set.

Pf WTS: $\langle u_i, u_j \rangle = 0$ for $i \neq j$.

Know: $\langle Tu_i, u_j \rangle = \langle u_i, Tu_j \rangle$ [Hermitian]

[eigen] $\parallel$

$\parallel$

$$\langle \lambda_i u_i, u_j \rangle$$

$$\langle u_i, \lambda_j u_j \rangle$$

$\parallel$

$\parallel$

$$\lambda_i \langle u_i, u_j \rangle$$

$$\overline{\lambda_j} \langle u_i, u_j \rangle$$

$$\text{so if } \langle u_i, u_j \rangle \neq 0 \Rightarrow \lambda_i = \overline{\lambda_j} \stackrel{\uparrow}{=} \lambda_j$$

eigenval of Hermitian
are real. $\square$

E.g. Let $\mathcal{H} = L^2(S^1)$, $D(f) = -if'$, $\Delta(f) = -f''$.

Claim $(e_n | n \in \mathbb{Z})$ is an eigenbasis of D and of Δ with eigenvalues $2\pi n$ and $4\pi^2 n^2$, resp.

Check $D(e_n) = (-i) 2\pi i n e^{2\pi i n x}$
 $= 2\pi n e_n$ ✓

$$\Delta(e_n) = D(D(e_n)) = 4\pi^2 n^2 e_n \quad \checkmark$$