

## Riemann $\zeta$ -values

Defn For  $s \in \mathbb{C}$  with  $\operatorname{Re}(s) > 1$ , define

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s};$$

this is the Riemann  $\zeta$ -function.

- Notes • Basic tools from complex analysis provide a unique meromorphic continuation of  $\zeta$  to  $\mathbb{C}$  (with simple pole at  $s=1$ ).
- Riemann hypothesis: If  $\zeta(s) = 0$  and  $0 < \operatorname{Re}(s) < 1$ , then  $\operatorname{Re}(s) = \frac{1}{2}$ .

Thm [Euler, 1737] For  $\operatorname{Re}(s) > 1$ ,

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

Pf Each term  $\frac{1}{1-p^{-s}}$  can be expanded as a geometric series

$1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \dots$ , which converges absolutely for  $\operatorname{Re}(s) > 1$ .

Note that  $\frac{1}{1-p^{-s}} \cdot \frac{1}{1-q^{-s}} = \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots\right) \left(1 + \frac{1}{q^s} + \frac{1}{q^{2s}} + \dots\right)$

$$= 1 + \frac{1}{p^s} + \frac{1}{q^s} + \frac{1}{(p^2)^s} + \frac{1}{(pq)^s} + \frac{1}{(q^2)^s} + \dots$$

$$= \sum_{n=p^a q^b} \frac{1}{n^s}.$$

Proceed by induction on max size of prime factors to get

$$\prod_{p \text{ prime}} \frac{1}{1-p^{-s}} = \zeta(s). \quad \square$$

Cor There are infinitely many primes.

Pf  $\infty = 1 + \frac{1}{2} + \frac{1}{3} + \dots = \zeta(1) = \prod_p \frac{1}{1 - \frac{1}{p}} = \prod_p \frac{p}{p-1}$   $\square$

$\lim_{N \rightarrow \infty} P_r(s \text{ int's from } \{1, \dots, N\} \text{ being coprime})$

Cor The asymptotic probability that  $s$  randomly selected positive integers share no common factors  $> 1$  is  $\frac{1}{\zeta(s)}$ .

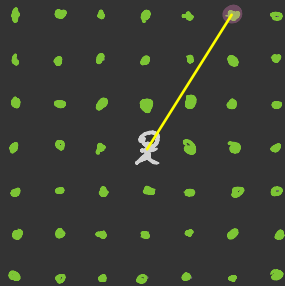
Pf We have  $\frac{1}{\zeta(s)} = \left( \prod_p \frac{1}{1 - \frac{1}{p^s}} \right)^{-1} = \prod_p \left( 1 - \frac{1}{p^s} \right)$ . Now

$\frac{1}{p^s} = \text{Prob}(s \text{ pos integers all divisible by } p)$  (Why?)

so  $1 - \frac{1}{p^s} = \text{Prob}(\text{at least one of } s \text{ positive integers not div by } p)$

By independence,  $\frac{1}{\zeta(s)} = \text{Prob}(s \text{ pos integers share no prime factor})$   $\square$

Cor Suppose trees are planted in a square grid and you are standing at  $(0,0)$ . Pick a tree at random. The probability that your view of the tree is not obstructed by another tree is  $\frac{1}{5(2)}$ .  $\square$



Goal Compute  $\zeta(s)$  for  $s \in \mathbb{Z}_{\geq 2}$

We will fail! We'll get  $\zeta(2s)$ ,  $s \geq 1$  integer  
 $\zeta(2s+1)$  — unknown!

Bernoulli polynomials

Today's convention: define functions on  $[0,1)$  then extend periodically

Inductive Defn  $B_1(x) = x - \frac{1}{2}$

$$\frac{d}{dx} B_k(x) = B_{k-1}(x)$$

$$\int_0^1 B_k(x) dx = 0$$

Q1 Are the  $B_k(x)$  well-defined polynomials?

Q2 Determine  $B_2(x)$ .

Let's compute  $\hat{B}_k(n) = \int_0^1 \overbrace{B_k(x)}^u \overbrace{e^{-2\pi i n x}}^{dv} dx$

$$= B_k(x) \frac{e^{-2\pi i n x}}{-2\pi i n} \Big|_0^1 - \int_0^1 \frac{e^{-2\pi i n x}}{-2\pi i n} B_{k-1}(x) dx$$

$$= \frac{1}{-2\pi i n} B_k(x) \Big|_0^1 + \frac{1}{2\pi i n} \hat{B}_{k-1}(n)$$

$$= \frac{1}{-2\pi i n} \int_0^1 B_k'(x) dx + \frac{1}{2\pi i n} \hat{B}_{k-1}(n)$$

$$= \frac{1}{2\pi i n} \hat{B}_{k-1}(n)$$

Since  $\hat{B}_1(n) = \int_0^1 (x - \frac{1}{2}) e^{-2\pi i n x} dx = \frac{-1}{2\pi i n}$  (comp'n)

we learn that

$$\hat{B}_k(n) = \frac{-1}{(2\pi i n)^k}$$

Thus the Fourier series of  $B_k(x)$  is

$$\frac{-1}{(2\pi i)^k} \sum_{0 \neq n \in \mathbb{Z}} \frac{e^{2\pi i n x}}{n^k}$$

Claim For  $k > 1$ ,  $B_k(0) = B_k(1) \Rightarrow B_k$  cts on  $S^1$  and

the Fourier series converges pointwise.

Pf  $B_k(1) - B_k(0) = \int_0^1 B_{k-1}(x) dx = 0$  for  $k > 1$  + IOU  $\square$  of cts

ptwise convergence  
✓ of Fourier  
series

Thm  $B_{2s}(0) = \frac{-2\zeta(2s)}{(2\pi i)^{2s}}$  for  $s \geq 1$  and

$B_{2s+1}(0) = 0$  for  $2s+1 > 1$ . ~ cancelling  $\pm n$  terms

In particular,

$$\zeta(2s) = (-1)^{s+1} 2^{2s-1} \pi^{2s} B_{2s}(0).$$

Prop The polynomials  $B_k(x)$  have generating function

$$1 + t B_1(x) + t^2 B_2(x) + t^3 B_3(x) + \dots = \frac{te^{tx}}{e^t - 1} \in \mathbb{C}[[t, x]]$$

pf Define  $f(t, x) = \text{LHS}$ . Then

$$\frac{\partial}{\partial x} f(t, x) = t \cdot 1 + t^2 B_1(x) + t^3 B_2(x) + \dots = t \cdot f(t, x).$$

Thus  $f(t, x) = C(t) \cdot e^{tx}$  for some  $C(t) \in \mathbb{C}[[t]]$ .

To compute  $C(t)$ , observe

$$\begin{aligned} \int_0^1 f(t, x) dx &= \int_0^1 1 dt + \sum_{l \geq 1} t^l \int_0^1 B_l(x) dx \\ &= 1 \end{aligned}$$

$$\text{while } \int_0^1 C(t) e^{tx} dx = C(t) \int_0^1 e^{tx} dx = C(t) \frac{e^{tx}}{t} \Big|_{x=0}^{x=1} = C(t) \frac{e^t - 1}{t}.$$

$$\text{Thus } C(t) = \frac{t}{e^t - 1} \quad \text{and } f(t, x) = \frac{t e^{tx}}{e^t - 1}. \quad \square$$

Evaluating at  $x=0$  gives

$$1 + t B_1(0) + t^2 B_2(0) + t^3 B_3(0) + \dots = \frac{t}{e^t - 1}$$

$$\Rightarrow \underbrace{1 - \frac{t}{2} - \frac{t}{e^t - 1}} = t^2 B_2(0) + t^4 B_4(0) + t^6 B_6(0) + \dots$$

Taylor expand:  $\frac{-t^2}{12} + \frac{t^4}{720} - \frac{t^6}{30240} + \frac{t^8}{1209600} - \dots$

$$\text{Hence } B_2(0) = \frac{-1}{12} \Rightarrow \zeta(2) = \frac{\pi^2}{6} \quad \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$$

$$B_4(0) = \frac{1}{720} \Rightarrow \zeta(4) = \frac{\pi^4}{90}$$

$$B_6(0) = \frac{-1}{30240} \Rightarrow \zeta(6) = \frac{\pi^6}{945}$$

$\approx \text{Prob}(\text{2 pos ints being coprime})$   
 $\approx 60.7\%$