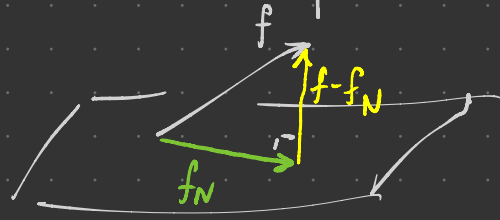


Bessel's inequality $\|f_N\| \leq \|f\|$.

PF Recall $f_N = \sum_{n=-N}^N \hat{f}(n) e_n = \text{proj}_{F_N} f$ where

$F_N = \text{span}\{e_{-N}, \dots, e_N\} \subseteq L^2(S')$ is the subspace of degree N trigonometric polynomials.

We have $f - f_N \perp f_N$



so by Pythagoras

$$\|f\|^2 = \|f - f_N + f_N\|^2 = \underbrace{\|f - f_N\|^2}_{\geq 0} + \|f_N\|^2 \geq \|f_N\|^2.$$

□

Defn The Dirichlet kernel $\{D_N \mid N \in \mathbb{N}\}$ is

$$D_N(x) := \sum_{n=-N}^N e_n(x).$$

The Fajér kernel $\{F_N \mid N \geq 1\}$ is

$$F_N(x) = \frac{1}{N} \sum_{k=0}^{N-1} D_k(x) \quad (\text{see demo})$$

Thm For $f \in C^0(S^1)$, $f * D_N = f_N$ and

$$f * F_N = \frac{1}{N} \sum_{k=0}^{N-1} f_k.$$

Pf By linearity of convolution, it suffices to show that

$$f * e_n = \hat{f}(n) e_n, \text{ and indeed}$$

$$\begin{aligned} (f * e_n)(x) &= \int_0^1 f(t) e_n(x-t) dt \\ &= \int_0^1 f(t) e_{-n}(t) e_n(x) dt \\ &= \langle f, e_n \rangle e_n(x) \\ &= \hat{f}(n) e_n(x). \quad \square \end{aligned}$$

Defn The N -th Cesàro sum of the Fourier series of f is

$$s_N(x) := (f * F_N)(x).$$

Lemma For $x \in S'$, $n \geq 0$, and $N \geq 1$, we have

$$D_n(x) = \begin{cases} \sin((2n+1)\pi x) / \sin(\pi x) & \text{if } x \neq 0 \\ 2n+1 & \text{if } x = 0 \end{cases}$$

$$F_N(x) = \begin{cases} \frac{1}{N} \sin^2(N\pi x) / \sin^2(\pi x) & \text{if } x \neq 0 \\ N & \text{if } x = 0 \end{cases}$$

pf Let $q = e^{\pi i x}$. Then

$$D_n(x) = \sum_{k=-n}^n q^{2k} = \begin{cases} (q^{-2n} - q^{2n+2}) / (1 - q^2) & \text{if } x \neq 0 \\ 2n+1 & \text{if } x = 0 \end{cases}$$

$$\text{Now } \frac{q^{-2n} - q^{2n+2}}{1 - q^2} = \frac{q^{2n+1} - q^{-2n-1}}{q - q^{-1}}$$

and $q^k - q^{-k} = 2i \sin(k\pi x)$, giving the result for $D_n(x)$.

$$\text{Now } F_N(0) = \frac{1}{N} \sum_{k=0}^{N-1} (2k+1) = \frac{N^2}{N} = N.$$

For $x \neq 0$,

$$\begin{aligned} F_N(x) &= \frac{1}{N} \frac{1}{1 - q^2} \left(\sum_{n=0}^{N-1} q^{-2n} - \sum_{n=0}^{N-1} q^{2n+2} \right) \\ &= \frac{1}{N} \frac{1}{1 - q^2} \left(\frac{1 - q^{-2N}}{1 - q^{-2}} - \frac{q^2 - q^{2N+2}}{1 - q^2} \right) \end{aligned}$$

$$= \frac{1}{N} \frac{1}{1-q^2} \left(\frac{1-q^{-2N}}{1-q^{-2}} + \frac{1-q^{2N}}{1-q^{-2}} \right)$$

$$= \frac{1}{N} \left(\frac{-q^{-2N} + 2 - q^{2N}}{-q^2 + 2 - q^{-2}} \right)$$

$$= \frac{1}{N} \left(\frac{q^N - q^{-N}}{q - q^{-1}} \right)^2$$

$$= \frac{1}{N} \frac{\sin^2(N\pi x)}{\sin^2(\pi x)} \quad \square$$

Thm The Fejér kernel F_N is a Dirac kernel.

Pf The expression
$$F_N(x) = \begin{cases} \frac{1}{N} \sin^2(N\pi x) / \sin^2(\pi x) & \text{if } x \neq 0 \\ N & \text{if } x = 0 \end{cases}$$

demonstrates $F_N \geq 0$. To show $\int_{-1/2}^{1/2} F_N(x) dx = 1$, it suffices

to show $\int_{-1/2}^{1/2} D_n(x) dx = 1$, which is HW.

To show $\lim_{N \rightarrow \infty} \int_{\delta \leq |x| \leq \frac{1}{2}} F_N(x) dx = 0$, first prove $F_N(x) \leq \frac{1}{N} \frac{1}{\sin^2(\pi\delta)}$

for $\delta \leq |x| \leq \frac{1}{2}$ (also HW). $\square$

Thm If $\{K_N\}$ is a Dirac kernel and $f \in C^0(S')$, then
 $(f * K_N)(x) \xrightarrow{N \rightarrow \infty} f(x)$ uniformly on S' .

Pf Sketch First prove

Lemma 1 $\forall \varepsilon_1 > 0 \exists 0 < \delta_1(\varepsilon_1) < \frac{1}{2}$ s.t. $\forall 0 < \delta < \delta_1(\varepsilon_1), x \in S', n \in \mathbb{N}$

$$\int_{-\delta}^{\delta} |f(x-t) - f(x)| |K_n(t)| dt < \varepsilon_1.$$

Then prove

Lemma 2 $\forall \delta, \varepsilon_2 > 0 \exists N_2(\delta, \varepsilon_2) \in \mathbb{N}$ s.t. $\forall n > N_2(\delta, \varepsilon_2), x \in S'$

$$\int_{\delta \leq |t| \leq \frac{1}{2}} |f(x-t) - f(x)| |K_n(t)| dt < \varepsilon_2.$$

Now show that $\forall \varepsilon > 0 \exists N(f, \varepsilon) \in \mathbb{N}$ s.t. $\forall x \in S', n \in \mathbb{Z}$,
if $n > N(f, \varepsilon)$, then $|(f * K_n)(x) - f(x)| < \varepsilon$ — which is
uniform convergence! $\square$

Cor For $f \in C^0(S')$, the Cesàro sum

$$S_N(x) = (f * F_N)(x) = \frac{1}{N} \sum_{n=0}^{N-1} f_n(x)$$

converges uniformly to $f(x)$ on S' as $N \rightarrow \infty$.

Pf F_N is a Dirac kernel, so the thm applies. $\square$

Inversion Thm For $f \in L^2(S')$, f_N = N-th Fourier polynomial of f ,

$$\lim_{N \rightarrow \infty} \|f - f_N\| = 0.$$

Pf Suppose $f \in L^2(S')$ and $\varepsilon > 0$. Since $C^0(S')$ is dense in $L^2(S')$,
 $\exists g \in C^0(S')$ with $\|f - g\| < \frac{\varepsilon}{2}$. Since $s_N[g] \rightarrow g$ uniformly,
it also converges to g in L^2 norm. Thus $\exists M \in \mathbb{N}$ s.t.

$$\|g - s_M\| < \frac{\varepsilon}{2}.$$

By Δ ineq,

$$\|f - s_M\| \leq \|f - g\| + \|g - s_M\| < \varepsilon.$$

By Best Approx'n,

$$\|f - f_M\| \leq \|f - s_M\| < \varepsilon.$$

trig poly
↓
[S]

Since Fourier polynomials give better approx's as $N \rightarrow \infty$,

$$\text{if } N > M, \quad \|f - f_N\| \leq \|f - f_M\| < \varepsilon. \quad \square$$

Cor $(e_n)_{n \in \mathbb{Z}}$ is an orthonormal basis of $L^2(S^1) \cong \ell^2(\mathbb{Z})$
 $f \mapsto (\hat{f}(n))_{n \in \mathbb{Z}}$ *isometry* □

Cor For $f, g \in L^2(S^1)$, TFAE: *the following are equivalent*
Lebesgue

① $f = g$ a.e. on S^1

② $\hat{f}(n) = \hat{g}(n) \quad \forall n \in \mathbb{Z}.$ □