2025. II. 3

Fourier series in $L^2(S^1)$

$\mathcal{B} = (e_n)_{n\in\mathbb{Z}}$ for $e_n : x \longmapsto e^{2\pi i n x} \in L^2(S^1)$

$S^1 = \mathbb{R}/\mathbb{Z}$

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "f"] \arrow[d] & \mathbb{C} \\
\mathbb{R}/\mathbb{Z} \arrow[ur, dashed] &
\end{tikzcd}$$

$\exists!$ extension iff $f(x) = f(x+1)$ $\forall x \in \mathbb{R}$

Q Why is $e_n \in L^2(S^1)$?

Q Why is $\mathcal{B}$ an orthonormal system?

$$\int_0^1 |e_n|^2 = \int_0^1 |e_n^2| = \int_0^1 |e_{2n}| = \int_0^1 1 = 1 < \infty$$

- $\|e_n\| = 1$ by
- $\langle e_n, e_m \rangle = \int_0^1 e_n \overline{e_m} = \int_0^1 e_{n-m} = \cdots = 0$

For $f \in L^2(S^1)$, the n-th Fourier coefficient of f is

$$\hat{f}(n) = \langle f, e_n \rangle$$

$$= \int_0^1 f(x) \overline{e_n(x)} \, dx$$

$$= \int_0^1 f(x) e^{-2\pi i n x} \, dx \in \mathbb{C}$$

The N-th Fourier polynomial of f is the projection of f onto $\operatorname{span}_{\mathbb{C}}\{e_{-N}, \ldots, e_N\}$, i.e

$$f_N(x) := \sum_{n=-N}^{N} \hat{f}(n) e_n(x).$$

The Fourier series of f is

$$\mathcal{F}[f](x) := \lim_{N\to\infty} f_N(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e_n(x).$$

inner prod space

Facts
- Call $\operatorname{span}_{\mathbb{C}}\{e_{-N},\dots,e_N\}$ the space of trigonometric polynomials of degree N. For $W \le V$, $\operatorname{proj}_W v$ is the vector in W closest to v: $\operatorname{proj}_W v = \operatorname{argmin}_{w\in W} \|v-w\|$.

  Thus $f_N(x)$ is the trig poly of degree N best approximating f in L^2 norm. (Best Approx'n Thm)

$$\|f_N\|^2 = \sum_{n=-N}^{N} |\hat{f}(n)|^2$$

monotonically increasing seq

Bessel's inequality $\|f_N\| \le \|f\|$ IOU.

- By the proof of our main theorem from Friday, $\sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$ converges so, in particular, we have the Riemann-Lebesgue lemma $\lim_{n\to\pm\infty} \hat{f}(n) = 0$.

- If $\mathcal{B}$ is an orthonormal basis, also get Parseval's identity: $\|f\|^2 = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$.

Forthcoming goal:

Inversion Thm $\mathcal{B}$ is an orthonormal basis for $L^2(S^1)$.

Cor $f = \mathcal{F}[f]$ in $L^2(S^1)$, and
$$L^2(S^1) \cong \ell^2(\mathbb{Z})$$
$$f \longmapsto (\hat{f}(n))_{n\in\mathbb{Z}}.$$

Equality in L^2 is not pointwise convergence.

Later we will show $\mathcal{F}[f]$ converges abolutely & uniformly to f for $f \in C^1(S^1)$ — and something similar holds for $C^0(S^1)$!

not an algebraic kernel

Convolutions & kernels

Fourier series have a useful relationship with an operation called convolution:

Fourier series : conv'n :: rings : mult'n

Defn For $f, g \in C^0(S^1)$, the convolution $f * g \in C^0(S^1)$ is given by

$$(f*g)(x) := \int_0^1 f(x-t)g(t)\,dt.$$

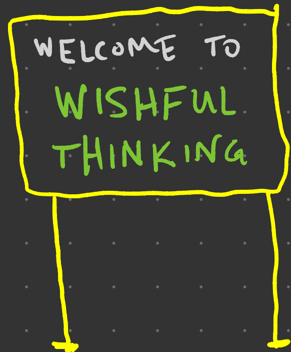
Moral exc $\quad t \mapsto f(x-t)\, g(t) \in C^0(S^1)$, $f * g \in C^0(S^1)$.

Thm
- Convolution is linear in each variable
- Convolution is commutative $\quad f*g = g*f$
- Convolution is associative $\quad (f*g)*h = f*(g*h)$
- For $f \in C^1(S^1)$, $g \in C^0(S^1)$, have $f*g \in C^1(S^1)$ and

$$(f*g)' = f' * g.$$

- For $f, g \in C^0(S^1)$, $\widehat{f*g}(n) = \hat{f}(n)\, \hat{g}(n)$.

Pf HW. $\square$



Suppose we have a Dirac delta function δ such that for $f \in C^0(S^1)$,

$$\int_{-1/2}^{1/2} \delta(x) f(x)\, dx = f(0).$$

Then $f * \delta(x) = \displaystyle\int_{-1/2}^{1/2} f(x-t)\, \delta(t)\, dt = f(x-0) = f(x)$

$\Rightarrow \hat{f}(n)\, \hat{\delta}(n) = \hat{f}(n) \Rightarrow \hat{\delta}(n) = 1$

i.e. $f * \delta = f$ — δ is an identity for the convolution op'n.

Now suppose we want to show $\lim_{n\to\infty} f_n = f$, and we have K_n such that $f * K_n = f_n$ and $\lim_{n\to\infty} K_n = \delta$. Then

$$\lim_{n\to\infty} f_n = \lim_{n\to\infty} f * K_n = f * \lim_{n\to\infty} K_n = f * \delta = f.$$

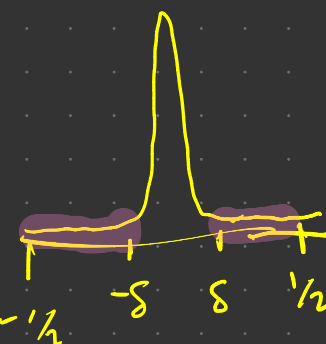
Defn A Dirac kernel on S^1 is a sequence of continuous

$K_n : [-\tfrac{1}{2}, \tfrac{1}{2}] \longrightarrow \mathbb{R}$ such that

(1) $K_n \geq 0$

(2) $\displaystyle\int_{-1/2}^{1/2} K_n(x)\,dx = 1.$

(3) For any $\delta > 0$, $\displaystyle\lim_{n\to\infty} \int_{\delta \leq |x| \leq 1/2} K_n(x)\,dx = 0$



Q What do we learn about $\displaystyle\int_{-\delta}^{\delta} K_n(x)\,dx$? $\xrightarrow[\delta \searrow 0,\ n\to\infty]{} 1$

Defn The Dirichlet kernel $\{D_N \mid N \in \mathbb{N}\}$ is

$$D_N(x) := \sum_{n=-N}^{N} e_n(x).$$

The Fejér kernel $\{F_N \mid N \geq 1\}$ is

$$F_N(x) = \frac{1}{N} \sum_{k=0}^{N-1} D_k(x).$$

(see demo)

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Thm For $f \in C^0(S^1)$, $f * D_N = f_N$ and

$$f * F_N = \frac{1}{N} \sum_{k=0}^{N-1} f_k.$$

Pf By linearity of convolution, it suffices to show that