

(Everything you should have learned in Math 201 about)

Inner product spaces (and now will learn really well)

An inner product space $(V, \langle \cdot, \cdot \rangle)$ is a $\mathbb{C}$ -vector space V equipped with a Hermitian form $\langle \cdot, \cdot \rangle$ that is positive definite.

- linear in first variable
- conjugate symmetric

- $\langle x, x \rangle \in \mathbb{R}_{\geq 0}$ with
 $\langle x, x \rangle = 0$ iff $x = 0$

E.g. $V = \mathbb{C}^n$, $\langle v, w \rangle = v^T \bar{w} = (v_1, v_2, \dots, v_n)$

Defn The norm of $v \in V$ is $\|v\| = \sqrt{\langle v, v \rangle}$

$$\left. \begin{array}{l} \langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C} \\ \bullet \langle \lambda u + v, w \rangle \\ \quad = \lambda \langle u, w \rangle + \langle v, w \rangle \\ \bullet \langle u, v \rangle = \overline{\langle v, u \rangle} \end{array} \right\} \begin{array}{l} \vdots \\ \vdots \\ \vdots \end{array}$$

Cauchy-Schwartz If V is an inner product space, then

$$\forall v, w \in V, \quad |\langle v, w \rangle| \leq \|v\| \|w\|.$$

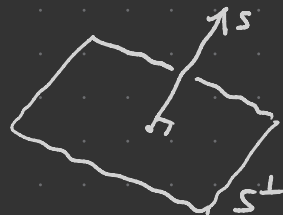
Cor • $\|\lambda v\| = |\lambda| \|v\| \quad \forall \lambda \in \mathbb{C}, v \in V$ (from sesqui-linearity)

• $\|v+w\| \leq \|v\| + \|w\|$ (triangle inequality)

Recall v is orthogonal to w when $\langle v, w \rangle = 0$; in this case write $v \perp w$.

For $S \subseteq V$, the orthogonal complement of S is

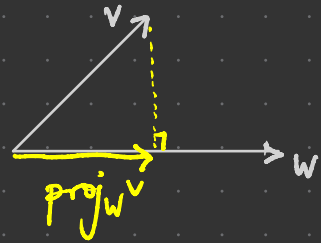
$$S^\perp = \{v \in V \mid v \perp s \quad \forall s \in S\}$$



The orthogonal projection of v onto w is

$$\text{proj}_w v := \frac{\langle v, w \rangle}{\|w\|^2} w$$

component of v along w



Defn A complete system in an inner product space V is a family $(a_j)_{j \in J}$ of vectors in V such that $\{a_j\}_{j \in J}^\perp = \{0\}$.

Call V separable when it contains a countable complete system.

formally: $a: J \rightarrow V$

trivial
subspace
 $\{0\} \leq V$

- To have $S^\perp = 0$ means that

$$\{v \in V \mid \langle v, s \rangle = 0 \ \forall s \in S\} = 0$$

$$\Leftrightarrow \text{if } \langle v, s \rangle = 0 \ \forall s \in S \text{ then } v = 0.$$

- Note $(v)_{v \in V}$ is always a complete system
separability demands a "small-ish" complete system.

E.g. Bases are complete systems.

Note $\mathbb{C}^n = \{f: \{1, 2, \dots, n\} \rightarrow \mathbb{C}\}$

E.g. Let $l^2(\mathbb{N}) = \{ f: \mathbb{N} \rightarrow \mathbb{C} \mid \sum_n |f(n)|^2 < \infty \}$

with $\langle f, g \rangle = \sum_n f(n) \overline{g(n)}$. For $j \in \mathbb{N}$, define $e_j \in l^2(\mathbb{N})$

by $e_j(n) = \begin{cases} 1 & \text{if } n=j \\ 0 & \text{if } n \neq j \end{cases}$

These $e_j \neq$
 $x \mapsto e^{2\pi i j x}$

Since $\langle f, e_j \rangle = \sum f(n) \overline{e_j(n)} = f(j)$

we learn that $(e_j)_{j \in \mathbb{N}}$ is a complete system
and $l^2(\mathbb{N})$ is separable.

Defn. An orthonormal system in an inner product space V

is a family $(h_j)_{j \in J}$ of vectors in V such that

$$\langle h_j, h_k \rangle = \delta_{jk} = \begin{cases} 1 & j=k \\ 0 & j \neq k \end{cases}$$

Kronecker delta

$$\begin{aligned} \text{Note } \|h_j\| &= \sqrt{\langle h_j, h_j \rangle} \\ &= \sqrt{1} = 1 \end{aligned}$$

An orthonormal system that is complete is called an orthonormal basis.

E.g. $(e_j)_{j \in \mathbb{N}}$ is an orthonormal basis of $\ell^2(\mathbb{N})$.

Prop Every separable inner product space admits an orthonormal basis.

pf Let $(a_j)_{j \in \mathbb{N}}$ be a complete system and apply Gram-Schmidt:

WLOG, every finite subset of a_j is lin ind.

without
loss of
generality

$$e_0 := \frac{a_0}{\|a_0\|}$$

$\vdots$

$$e'_{k+1} := a_{k+1} - \sum_{j=0}^k \underbrace{\langle a_{k+1}, e_j \rangle}_{\text{proj}_{e_j} a_{k+1}} e_j$$

$$e_{k+1} := \frac{e'_{k+1}}{\|e'_{k+1}\|}$$

$\vdots$

Q Why isn't $e'_{k+1} = 0$?

A By lin ind of a_j

If V is finite dim, this terminates to produce a basis.

If V is infinite diml, get a sequence $(e_j)_{j \in \mathbb{N}}$ that is orthonormal.

Suppose $\langle h, e_j \rangle = 0 \ \forall j \in \mathbb{N}$. Then $\langle h, a_j \rangle = 0 \ \forall j \in \mathbb{N}$, so $h = 0$, so $(e_j)_j$ is complete as well. $\square$

Then Suppose V is an infinite-dimensional Hilbert space with orthonormal basis (e_j) . Then every element $v \in V$ can be uniquely expressed as

$$v = \sum_{j \in \mathbb{N}} c_j e_j$$

• complete inner prod space: Cauchy sequences converge

A Cauchy sequence in V is $(v_i)_{i \in \mathbb{N}}$

such that $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ s.t.

if $i, j > N$, then $\|v_i - v_j\| < \varepsilon$.

In general, (v_i) convergent $\Rightarrow$ Cauchy.

Completeness is converse.

$\exists v \in V$ s.t.

$\forall \varepsilon > 0 \exists N$ s.t.

if $i > N$ then $\|v_i - v\| < \varepsilon$

in this case, say that

$\lim v_i = v$ (in $\|\cdot\|$)

with the sum convergent in V , and c_j satisfying

$$\sum_{j \in \mathbb{N}} |c_j|^2 < \infty.$$

In fact, $c_j = \langle v, e_j \rangle$ and $v \mapsto (\langle v, e_j \rangle)_{j \in \mathbb{N}}$ is an

isometry $V \longrightarrow \ell^2(\mathbb{N})$

In particular,

$$\langle v, v' \rangle = \sum_{j \in \mathbb{N}} \langle v, e_j \rangle \overline{\langle v', e_j \rangle}$$

$$\text{and } \|v\|^2 = \sum_{j \in \mathbb{N}} |\langle v, e_j \rangle|^2.$$

- linear trans'n $V \xrightarrow{T} W$ that preserves inner products:

$$\langle T v, T v' \rangle = \langle v, v' \rangle$$

- necessarily injective

- if surjective, then admits isometric inverse, called unitary