MATH 372: COMBINATORICS
HOMEWORK 04

DUE FRIDAY, 30 OCTOBER, 10P.M., VIA GRADESCOPE

Remark. **When you can start.** Problem 1 uses the constructions from Monday, October 12, and Problem 2 uses the multisets and partitions from Wednesday, October 14: do these two before or over fall break. Problem 3 needs Monday, October 26 (Young diagrams and q-binomial coefficients, which is also the break reading), and Problem 4 is best done after Wednesday, October 28. The second half is deliberately short.

Remark. **Conventions.** A *composition* of n is a finite sequence of positive integers with sum n; a *partition* of n is a multiset of positive integers with sum n, written as a weakly decreasing sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$. The empty composition and the empty partition are the unique objects of size 0. Fibonacci numbers are $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$. For $0 \leq k \leq N$, let $L(k, N-k)$ be the set of partitions with at most k parts, each at most $N-k$ (the Young diagrams that fit in a box with k rows and $N-k$ columns), ordered by containment of diagrams, and define

$$\begin{bmatrix} N \\ k \end{bmatrix}_q = \sum_{\lambda \in L(k,N-k)} q^{|\lambda|}, \qquad |\lambda| = \lambda_1 + \lambda_2 + \cdots .$$

You may use the recurrence from class on October 26, $\begin{bmatrix} N \\ k \end{bmatrix}_q = \begin{bmatrix} N-1 \\ k \end{bmatrix}_q + q^{N-k} \begin{bmatrix} N-1 \\ k-1 \end{bmatrix}_q$ for $0 < k < N$, together with $\begin{bmatrix} N \\ 0 \end{bmatrix}_q = \begin{bmatrix} N \\ N \end{bmatrix}_q = 1$.

Remark. **On write-ups.** Each problem is worth up to five points for mathematical content and up to two for writing. Write for another student in this course: say what you are going to show before you show it, justify each step, and use complete sentences. Whenever you give a specification, say in words what the objects are before writing the generating function. End your submission with a short *Methods and tools* note: what software you used and for what, and how you checked anything a machine produced for you.

Problem 1 (specifications). (a) Let $\mathcal{A}$ be the class of binary words with no three consecutive 1s, size = length. Give a specification of $\mathcal{A}$ in terms of SEQ, $+$ and $\times$, deduce its generating function $A(z)$ as a single fraction, and compute $a_0, \ldots, a_6$ from it. Check a_4 by listing the words of length 4 that are *not* in $\mathcal{A}$.

(b) Let $\mathcal{C}$ be the class of compositions in which every part is at least 2. Give a specification, find $C(z)$, and show that for $n \geq 2$ the number c_n equals the number of compositions of $n-2$ into parts 1 and 2. Then find a bijection that explains this. (Suggestion: rewrite each part $a \geq 2$ of a composition in $\mathcal{C}$ as 2 followed by $a-2$ ones.)

(c) Mark each part of a composition with a variable u: show that

$$\sum_{n,k \geq 0} \#\{\text{compositions of } n \text{ with exactly } k \text{ parts}\}\, u^k z^n = \frac{1}{1 - \dfrac{uz}{1-z}},$$

and deduce that the number of compositions of $n \geq 1$ with exactly k parts is $\binom{n-1}{k-1}$. Explain the same count with the "dots and bars" picture from class, and check that summing over k gives the total you already know.

Problem 2 (partitions). (a) Prove, using generating functions, that for every n the number of partitions of n with no part divisible by 3 equals the number of partitions of n in which no part appears three or more times.

(b) Check (a) for $n = 6$ by listing both sets of partitions.

(c) State and prove the analogous statement with 3 replaced by any integer $d \geq 2$. What statement from class is the case $d = 2$?

Problem 3 (q-binomial coefficients). (a) Prove the product formula

$$\begin{bmatrix} N \\ k \end{bmatrix}_q = \frac{(1-q^N)(1-q^{N-1})\cdots(1-q^{N-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)},$$

by showing that the right-hand side satisfies the recurrence and boundary values in the Conventions. (In particular the right-hand side is a polynomial, which is not obvious.)

(b) Prove the second recurrence $\begin{bmatrix} N \\ k \end{bmatrix}_q = q^k \begin{bmatrix} N-1 \\ k \end{bmatrix}_q + \begin{bmatrix} N-1 \\ k-1 \end{bmatrix}_q$ by sorting the diagrams in $L(k, N-k)$ according to their first column.

(c) Show that the coefficients of $\begin{bmatrix} N \\ k \end{bmatrix}_q$ are symmetric: $q^{k(N-k)} \begin{bmatrix} N \\ k \end{bmatrix}_q \Big|_{q \to q^{-1}} = \begin{bmatrix} N \\ k \end{bmatrix}_q$. (Find an involution of $L(k, N-k)$.)

Problem 4 (unimodality with no linear algebra, for two rows). Fix $n \geq 1$ and consider the poset $L(2, n)$: partitions (λ_1, λ_2) with $n \geq \lambda_1 \geq \lambda_2 \geq 0$, graded by $\lambda_1 + \lambda_2$, of rank $2n$.

(a) Show that the number of elements of rank i is $\lfloor i/2 \rfloor + 1$ for $0 \leq i \leq n$, and use (3c) or a direct argument for $n \leq i \leq 2n$. Conclude that $L(2, n)$ is rank-symmetric and rank-unimodal, and write down its rank sizes when $n = 4$.

(b) Find order-matchings $L(2,n)_0 \to L(2,n)_1 \to \cdots \to L(2,n)_n$ and $L(2,n)_n \leftarrow L(2,n)_{n+1} \leftarrow \cdots \leftarrow L(2,n)_{2n}$, and verify that they are order-matchings. (Change one part at a time.)

(c) Conclude that the largest antichain in $L(2, n)$ has $\lfloor n/2 \rfloor + 1$ elements, and draw the chains your matchings produce when $n = 4$.

Bonus Problem (subspaces). Let q be a prime power and $\mathbb{F}_q$ the field with q elements. Show that the number of k-dimensional subspaces of $\mathbb{F}_q^N$ is $\begin{bmatrix} N \\ k \end{bmatrix}_q$ evaluated at that q. (Count ordered bases two ways.) How many 2-dimensional subspaces does $\mathbb{F}_2^4$ have?