MATH 372: COMBINATORICS
HOMEWORK 02

DUE FRIDAY, 25 SEPTEMBER, 10P.M., VIA GRADESCOPE

Remark. **When you can start.** Problems 1 and 2 use only what we have done through Monday, September 14. Problem 3 needs Wednesday the 16th, and Problem 4 needs Friday the 18th. The set is released early so that the first half can be done while the lab is also open; do not wait for Friday to begin.

Remark. **Conventions.** Graphs are finite and undirected. For $u, v \in (\mathbb{Z}/2)^n$ the dot product $u \cdot v = \sum_i u_i v_i$ is taken mod 2, $\omega(u)$ is the number of 1's in u, and $\chi_u \colon (\mathbb{Z}/2)^n \to \mathbb{R}$ is the function $\chi_u(v) = (-1)^{u \cdot v}$, as in class. A poset is finite; P_i denotes the ith level of a graded poset and $p_i = \#P_i$. An *order-matching* from P_i to P_{i+1} is a one-to-one map $\mu \colon P_i \to P_{i+1}$ with $x < \mu(x)$ for all x; from P_i to P_{i-1} it is a one-to-one map with $\mu(x) < x$.

Remark. **On these solutions.** Each problem is worth up to five points for mathematical content and up to two for writing. Write for another student in this course: say what you are going to show before you show it, justify each step, and use complete sentences. End your submission with a short *Methods and tools* note — three or four sentences on what software you used and for what, and how you checked anything a machine produced for you.

Problem 1 (the two-flip graph). Fix $n \geq 3$. Let G_n be the graph with vertex set $(\mathbb{Z}/2)^n$ in which u and v are adjacent exactly when they differ in *exactly two* coordinates. (So G_n is to the cube as "change two bits" is to "change one bit.")

(a) Show that G_n is regular and find its degree.
(b) Show that each χ_u is an eigenvector of the adjacency matrix of G_n, and find its eigenvalue as a function of $\omega(u)$. Then write out $\mathrm{spec}(G_4)$ completely, with multiplicities.
(c) How many connected components does G_n have? Describe them. (Which vertices can a walk starting at 0 reach?)
(d) Use $\mathrm{spec}(G_4)$ to compute the number of closed walks of length 3 in G_4, and deduce how many triangles G_4 contains. Then confirm that number by a direct count, using your description of the components in (c).

Problem 2 (coming home). Let G be a d-regular graph on p vertices with adjacency matrix A, and let $M = \frac{1}{d}A$, so that $(M^\ell)_{uv}$ is the probability that the simple random walk started at u is at v after ℓ steps. Write $p_\ell(u) = (M^\ell)_{uu}$ for the probability of being home after ℓ steps.

(a) Explain why $p_\ell(u)$ does not depend on u when G is the complete graph K_p, and also when G is the cube Q_n. Conclude that in both cases $p_\ell = \mathrm{tr}(M^\ell)/p$.
(b) For K_p, find p_ℓ exactly as a function of p and ℓ, and compute $\lim_{\ell \to \infty} p_\ell$.
(c) For Q_n, show that $p_\ell = 0$ when ℓ is odd, find $\lim_{m \to \infty} p_{2m}$, and explain in one sentence why the answer is 2^{1-n} rather than 2^{-n}.

Problem 3 (a poset where the matchings can be written down). Fix integers $a, b \geq 1$ and let P be the set of pairs (i, j) with $0 \leq i \leq a$ and $0 \leq j \leq b$, ordered by $(i, j) \leq (i', j')$ if and only if $i \leq i'$ and $j \leq j'$. (If you like: P is the set of divisors of $2^a 3^b$, ordered by divisibility.)

(a) Show that P is graded of rank $a+b$, with $P_k = \{(i,j) : i+j = k\}$. Compute the rank sizes $p_0, \dots, p_{a+b}$ and show the sequence is symmetric and unimodal.

(b) Assume $a \le b$. Find order-matchings

$$P_0 \to P_1 \to \cdots \to P_b \qquad \text{and} \qquad P_b \leftarrow P_{b+1} \leftarrow \cdots \leftarrow P_{a+b},$$

and verify that they are order-matchings. (There is a very simple choice; try changing only one coordinate.)

(c) Conclude that the largest antichain in P has exactly $\min(a,b)+1$ elements.

(d) Draw the chain decomposition of P that your matchings produce when $a=2$ and $b=3$.

Problem 4 (the linear algebra method). Recall the operators on the Boolean algebra B_n: for a set x with $|x| = i$,

$$U_i(x) = \sum_{\substack{|y|=i+1 \\ y \supset x}} y, \qquad D_i(x) = \sum_{\substack{|z|=i-1 \\ z \subset x}} z,$$

extended linearly to $U_i \colon \mathbb{R}(B_n)_i \to \mathbb{R}(B_n)_{i+1}$ and $D_i \colon \mathbb{R}(B_n)_i \to \mathbb{R}(B_n)_{i-1}$.

(a) For $n=3$, write out the matrices $[U_1]$ and $[D_2]$ with respect to the bases $(B_3)_1 = \{1,2,3\}$ and $(B_3)_2 = \{12, 13, 23\}$, in those orders. Then compute $[D_2][U_1] - [U_0][D_1]$ and check that it is what the identity from class predicts.

(b) Now let P be any graded poset and suppose $U \colon \mathbb{R}P_i \to \mathbb{R}P_{i+1}$ is linear, *order-raising* (each $U(x)$ is a linear combination of elements $y > x$), and *onto*. Prove that there is an order-matching from P_{i+1} to P_i.

In class we proved the companion statement — *one-to-one* order-raising gives a matching from P_i to P_{i+1} — by extracting a nonzero term from a determinant. Your proof should run the same way, but be careful about which rows and columns you are choosing and why.

Bonus Problem (LYM). Let $\mathcal{A}$ be an antichain in B_n. Show that

$$\sum_{A \in \mathcal{A}} \binom{n}{|A|}^{-1} \le 1,$$

and deduce Sperner's theorem from it. (Count, in two ways, the pairs $(\mathcal{C}, A)$ where $\mathcal{C}$ is a maximal chain $\emptyset \subset \{x_1\} \subset \{x_1, x_2\} \subset \cdots$ in B_n and $A \in \mathcal{A}$ lies on $\mathcal{C}$. How many maximal chains pass through a given A? How many can pass through two different members of an antichain?)