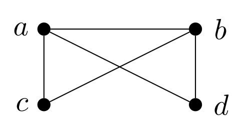
MATH 372: COMBINATORICS
HOMEWORK 01

DUE FRIDAY, 11 SEPTEMBER, 10P.M., VIA GRADESCOPE

Remark. **Conventions.** All graphs here are finite and undirected, and they are allowed to have loops. If G has vertex set $\{v_1, \ldots, v_p\}$, its *adjacency matrix* $A(G)$ is the $p \times p$ matrix whose (i, j) entry is the number of edges joining v_i to v_j; a loop at v_i contributes 1 to the (i, i) entry. A *walk of length* ℓ from u to v is a sequence $u = u_0, u_1, \ldots, u_\ell = v$ of vertices in which each consecutive pair is joined by an edge, and the walk is *closed* if $u = v$. Since $A(G)$ is a real symmetric matrix, its eigenvalues are real; the *spectrum* $\operatorname{spec}(G)$ is the multiset of those eigenvalues, listed with multiplicity. Throughout, $\ell \geq 1$.

Remark. **On these solutions.** Each problem is worth up to five points for mathematical content and up to two points for writing. Write your solutions as explanations addressed to another student in this course: say what you are going to show before you show it, justify each step, and use complete sentences even when symbols are doing most of the work. End your submission with a short *Methods and tools* note — three or four sentences saying what software you used and for what, and how you checked anything a machine produced for you.

Problem 1. Let D be the graph on four vertices pictured below, with adjacency matrix A taken with respect to the ordering a, b, c, d:



$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

(a) Determine $\operatorname{spec}(D)$.
(b) Determine the number of closed walks of length 4 in D in two genuinely different ways: once using your answer to (a), and once by a computation that does not use the eigenvalues at all. Confirm that the two answers agree.

The mathematics here is routine, and that is deliberate. This problem exists so that we can agree on what a written solution in this course looks like, so spend your effort on the exposition: a reader who has not seen D should be able to follow your work from beginning to end without supplying anything themselves.

Problem 2. Suppose G is a graph with 15 vertices, and suppose that for every $\ell \geq 1$ the number of closed walks of length ℓ in G is

$$8^\ell \;+\; 2 \cdot 3^\ell \;+\; 3 \cdot (-1)^\ell \;+\; (-6)^\ell \;+\; 5^\ell.$$

Let G' be the graph obtained from G by adding a loop at every vertex, in addition to whatever loops G already has. How many closed walks of length ℓ does G' have?

Give a complete justification, not merely the formula. (Linear-algebraic reasoning is the intended route. If you would also like to think about what the answer says combinatorially, you are welcome to, but it is not required.)

Problem 3. Let $r, s \geq 1$. The *complete bipartite graph* $K_{r,s}$ has vertices $u_1, \ldots, u_r, v_1, \ldots, v_s$, with exactly one edge joining u_i to v_j for every i and j, and no other edges. (So $K_{r,s}$ has $r + s$ vertices and rs edges.)

(a) By purely combinatorial reasoning — that is, by counting walks directly, without computing any eigenvalues — determine the number of closed walks of length ℓ in $K_{r,s}$.

(b) Deduce $\operatorname{spec}(K_{r,s})$ from your answer to (a).

Part (b) runs in the opposite direction from everything we did in class this week: there we computed a spectrum and read off walk counts, and here you are asked to compute walk counts and read off a spectrum. Say clearly what makes that deduction valid.

Problem 4. A graph G is *bipartite* if its vertex set can be written as a disjoint union $V = X \sqcup Y$ in such a way that every edge of G joins a vertex of X to a vertex of Y. (In particular a bipartite graph has no loops.)

Show that if G is bipartite, then its nonzero eigenvalues come in pairs $\pm\lambda$: that is, for each $\lambda \neq 0$, the eigenvalue λ and the eigenvalue $-\lambda$ occur in $\operatorname{spec}(G)$ with the same multiplicity.

Give a *walk-counting* argument. There is a slicker proof available using eigenvectors, and you are welcome to find it and include it as a remark, but the argument you submit should proceed by counting walks.

Bonus Problem. Let G be a graph and let Δ be the largest degree of any vertex of G, where the *degree* of v is the number of edges incident to v. Let λ_1 be the largest eigenvalue of $A(G)$. Show that $\lambda_1 \leq \Delta$.