Math 372

Ordinary generating functions
(Flajolet–Sedgewick, §I.1 and Appendix A.5)

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Fishing Expedition

5 min

Ontario: OEIS sequence.

1. What is the smallest case you checked by hand?
2. What would you need in order to prove the formula?

Reading check

What does $[z^n]$ mean?

$$[z^2]\,(1 + 3z + 5z^2 + 7z^3) = 5$$

$$[z^n]\,(1 + z)^m = \binom{m}{n}$$

$$[z^n]\,\frac{1}{1 - z} = 1$$

$[z^n]\,F(z)$ is the coefficient of z^n in $F(z)$.

The definition

Definition

A **combinatorial class** is a set $\mathcal{A}$ with a **size** function $|\cdot| : \mathcal{A} \to \{0, 1, 2, \dots\}$ such that each size occurs only finitely often.

Definition

Its **counting sequence** is $a_n = \#\{\alpha \in \mathcal{A} : |\alpha| = n\}$, and its **ordinary generating function** (OGF) is

$$A(z) = \sum_{n \geq 0} a_n z^n = \sum_{\alpha \in \mathcal{A}} z^{|\alpha|}.$$

The two sums are the same sum, grouped differently. $[z^n]A(z) = a_n$.

Example 1: binary words

$\mathcal{W}$ = binary words, size = length, ε = the empty word.

$$\begin{aligned} W(z) &= \sum_{w\in\mathcal{W}} z^{|w|} \\ &= \underbrace{z^{|\varepsilon|}}_{1} + \underbrace{z^{|0|}+z^{|1|}}_{2z} + \underbrace{z^{|00|}+z^{|01|}+z^{|10|}+z^{|11|}}_{4z^2} + \underbrace{z^{|000|}+\cdots+z^{|111|}}_{8z^3} + \cdots \\ &= \sum_{n\geq 0} 2^n z^n = \frac{1}{1-2z}. \end{aligned}$$

Warning

What does the last equal sign mean? What if $z = 1$? We will make sense by the end of class.

Example 2: permutations

$\mathcal{P}$ = permutations of $[n] = \{1, \ldots, n\}$, for all $n \geq 0$,

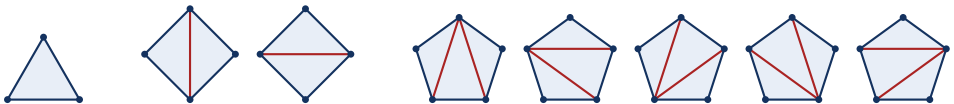
size = n, the number of objects permuted.

$$p_n = n!, \qquad P(z) = \sum_{n \geq 0} n!\, z^n = 1 + z + 2z^2 + 6z^3 + 24z^4 + 120z^5 + \cdots$$

- Converges for no $z \neq 0$.
- Later (Chapter III), we will be interested in implications of convergence, and hence, we will be motivated to find a different type of generating function for permutations (Chapter II).

Example 3: triangulations of a convex polygon

$\mathcal{T}$ = triangulations of a convex polygon, size = number of triangles.



$t_1 = 1$ $t_2 = 2$ $t_3 = 5$

$t_0 = 1$ (the empty triangulation of a 2-gon), and then $1, 1, 2, 5, 14, 42, \ldots$

See §I.1

$$T(z) = \frac{1 - \sqrt{1 - 4z}}{2z}$$

The first *non-rational* generating function of the course. The square root will matter in November.

Isomorphic classes

Definition

Two classes with the same counting sequence (equivalently, the same OGF) are **(combinatorially) isomorphic**, $\mathcal{A} \cong \mathcal{B}$.

Example: three isomorphic classes

Words of length n, subsets of $[n]$, compositions of $n+1$. Each has counting sequence 2^n and OGF given by $1/(1-2z)$.

Wednesday's closed walks

Fix a vertex v of a graph with adjacency matrix A. Let $\mathcal{K}$ be closed walks from v, size = length. Then $k_n = (A^n)_{vv}$, so $K(z) = \sum_{n\geq 0}(A^n)_{vv}\, z^n = \left[(I - zA)^{-1}\right]_{vv}$, a *rational* function: a quotient of polynomials, by Cramer's rule, with denominator $\det(I - zA)$.

The ring of formal power series $\mathbb{C}[[z]]$

Definition

A **formal power series** is a sequence $(f_0, f_1, f_2, \dots)$ with every $f_n \in \mathbb{C}$, written $F = \sum_{n \geq 0} f_n z^n$. Let $\mathbb{C}[[z]]$ denote the set of all formal power series.

Two series are *equal* when all their coefficients agree.

Definition

For $F = \sum f_n z^n$ and $G = \sum g_n z^n$ in $\mathbb{C}[[z]]$,

$$[z^n](F+G) = f_n + g_n, \qquad [z^n](FG) = \sum_{k=0}^{n} f_k \, g_{n-k}.$$

Fact

$\mathbb{C}[[z]]$ is a commutative ring with identity $1 = 1 + 0z + 0z^2 + \cdots$.

What the definitions say

Let $F = \sum f_n z^n$ and $G = \sum g_n z^n$.

$$\begin{aligned} F + G &= (f_0 + f_1 z + f_2 z^2 + \cdots) + (g_0 + g_1 z + g_2 z^2 + \cdots) \\ &= (f_0 + g_0) + (f_1 + g_1)\, z + (f_2 + g_2)\, z^2 + \cdots \end{aligned}$$

$$\begin{aligned} FG &= (f_0 + f_1 z + f_2 z^2 + f_3 z^3 + \cdots)(g_0 + g_1 z + g_2 z^2 + g_3 z^3 + \cdots) \\ &= f_0 g_0 + (f_0 g_1 + f_1 g_0)\, z + (f_0 g_2 + f_1 g_1 + f_2 g_0)\, z^2 \\ &\qquad + (f_0 g_3 + f_1 g_2 + f_2 g_1 + f_3 g_0)\, z^3 + \cdots \end{aligned}$$

The definitions are exactly what adding and multiplying as if they were polynomials gives. Each coefficient of $F + G$ is one addition, and each coefficient of FG is a sum of $n + 1$ products: **finite computations; no convergence is involved.**

Fact (i): inverses

Fact (i)

F has a multiplicative inverse in $\mathbb{C}[[z]]$ if and only if $f_0 \neq 0$.

Reason. We want G with $FG = 1$. Multiply out:

$$FG = f_0 g_0 + (f_0 g_1 + f_1 g_0)\, z + (f_0 g_2 + f_1 g_1 + f_2 g_0)\, z^2 + \cdots = 1 + 0z + 0z^2 + \cdots$$

Comparing coefficients: $f_0 g_0 = 1$, $f_0 g_1 + f_1 g_0 = 0$, $f_0 g_2 + f_1 g_1 + f_2 g_0 = 0$, $\ldots$

If $f_0 \neq 0$, solve them in order; each equation has exactly one new unknown:

$$g_0 = \frac{1}{f_0}, \qquad g_1 = -\frac{f_1 g_0}{f_0}, \qquad g_2 = -\frac{f_1 g_1 + f_2 g_0}{f_0}, \qquad g_n = -\frac{1}{f_0} \sum_{k=1}^{n} f_k\, g_{n-k}.$$

The recurrence always has a solution. If $f_0 = 0$, the first equation $f_0 g_0 = 1$ already fails.

The warning from Example 1, resolved (1 of 2)

Warning (Example 1)

$W(z) = 1 + 2z + 4z^2 + 8z^3 + \cdots = \frac{1}{1-2z}$. What does the last equal sign mean? What if $z = 1$?

Meaning. The notation $\frac{1}{1-2z}$ means *the multiplicative inverse of* $1 - 2z$ *in* $\mathbb{C}[[z]]$, which exists by Fact (i) since $f_0 = 1 \neq 0$.

Computing it. We want $G = g_0 + g_1 z + g_2 z^2 + \cdots$ with $(1 - 2z)\, G = 1$. Multiply out:

$$\begin{aligned}(1-2z)(g_0 + g_1 z + g_2 z^2 + g_3 z^3 + \cdots) &= g_0 + (g_1 - 2g_0)\, z + (g_2 - 2g_1)\, z^2 + \cdots \\ &= 1 + 0z + 0z^2 + \cdots\end{aligned}$$

Comparing coefficients: $g_0 = 1$, $g_1 = 2g_0 = 2$, $g_2 = 2g_1 = 4$, $\ldots$, so $g_n = 2^n$.

The warning from Example 1, resolved (2 of 2)

So the multiplicative inverse of $1-2z$ in $\mathbb{C}[[z]]$ is $1+2z+4z^2+8z^3+\cdots$.

Checking it.
$(1-2z)(1+2z+4z^2+8z^3+\cdots) = 1+(2-2)\,z+(4-4)\,z^2+(8-8)\,z^3+\cdots = 1.$

So

$1+2z+4z^2+\cdots$ *is* $\frac{1}{1-2z}$ in $\mathbb{C}[[z]]$, no convergence involved. Nothing is evaluated; setting $z=1$ is not considered in this ring.

Aside: so when does analysis matter?

Convergence, poles, growth enter only when we ask *how big* a_n is. That is Act III.

Fact (ii): composition

Fact (ii)

If $g_0 = 0$, then $F(G(z)) = \sum_{k\geq 0} f_k\, G(z)^k$ is defined in $\mathbb{C}[[z]]$.

Reason. Let $F(z) = \sum_{k\geq 0} f_k z^k$ and $G(z) = \sum_{m\geq 0} g_m z^m$.

$$F(G(z)) = \sum_{k\geq 0} f_k\, G(z)^k$$

$$[z^n]F(G(z)) = \sum_{k\geq 0} f_k\, [z^n]G(z)^k$$

If $g_0 \neq 0$: $[z^0]G^k = g_0^k$, so $[z^0]F(G(z)) = \sum_{k\geq 0} f_k g_0^k$ is potentially an infinite sum.

If $g_0 = 0$: $G = z\,(g_1 + g_2 z + \cdots)$, so $G^k = z^k(\cdots)$ has no terms below degree k. So for $k > n$, $[z^n]G^k = 0$, and $[z^n]F(G(z)) = \sum_{k=0}^{n} f_k\, [z^n]G(z)^k$, a finite sum.

Fact (iii): equality of OGFs

Fact (iii)

An equality of OGFs is a counting statement.

Let $\mathcal{A}$ and $\mathcal{B}$ be combinatorial classes with OGFs $A(z) = \sum a_n z^n$ and $B(z) = \sum b_n z^n$, respectively. Then $A(z) = B(z)$ means that for every n,

$$\#\{\alpha \in \mathcal{A} : |\alpha| = n\} \;=\; a_n \;=\; b_n \;=\; \#\{\beta \in \mathcal{B} : |\beta| = n\}.$$

The equality of series says: **for every n, the classes $\mathcal{A}$ and $\mathcal{B}$ have the same number of objects of size n.**

Group problems

14 min

Two people on (a), three on (b).

(a)

Let $\mathcal{F}$ be binary words with **no two consecutive 1s**, size = length. Compute $f_0, \ldots, f_5$ by hand, find a recurrence, and write $F(z)$ as a single fraction.

(b)

Let $\mathcal{C}$ be **compositions of n into parts equal to 1 or 2** (ordered sums, so $1+2$ and $2+1$ differ), size = n. Compute $c_0, \ldots, c_5$, find $C(z)$, and find the relation between $C(z)$ and $F(z)$ from (a) along with a bijection that explains it.

More group problems

(c)

Let $\mathcal{K}$ be closed walks from a fixed vertex of K_3, size = length, so $k_n = \frac{1}{3}(2^n + 2(-1)^n)$. Write $K(z)$ as a single fraction. Then compute the $(1,1)$ entry of $(I - zA)^{-1}$ directly, for $A = A(K_3) = J - I$, and confirm it is the same function.

(d)

Compute the first five coefficients of $1/P(z)$, where $P(z) = \sum n!\, z^n$. Look the sequence up.

(a) Words with no two consecutive 1s

$f_0, \ldots, f_5 = 1, 2, 3, 5, 8, 13$.

Recurrence $f_n = f_{n-1} + f_{n-2}$: a word ends in 0 (anything valid of length $n-1$, then 0) or in 01 (anything valid of length $n-2$, then 01). The lone word "1" is the base case.

Fraction. From the recurrence, $(1 - z - z^2)F(z) = f_0 + (f_1 - f_0)z = 1 + z$, so

$$F(z) = \frac{1+z}{1-z-z^2}.$$

Structural version: every such word is a sequence of blocks 0 and 10, then optionally a final 1: $\mathcal{F} = \mathrm{SEQ}(\{0, 10\}) \times (\{\varepsilon\} + \{1\})$, which reads directly as $\dfrac{1}{1-(z+z^2)} \cdot (1+z)$. (See Monday's reading.)

(b) Compositions into 1s and 2s

$c_0, \ldots, c_5 = 1, 1, 2, 3, 5, 8, \quad c_n = c_{n-1} + c_{n-2}$ (first part is 1 or 2),

$C(z) = \dfrac{1}{1 - z - z^2}$.

Relation. $F(z) = (1 + z)\, C(z)$, i.e. $f_n = c_n + c_{n-1}$.

Bijection 1, for $f_n = c_n + c_{n-1}$

A word with no 11 that **ends in 0** splits into blocks 0 and 10 of sizes 1 and 2, respectively, corresponding to a composition of n. (c_n of these.) A word that **ends in 1** has 0 or nothing before that 1; delete the 1 to get a word of length $n - 1$ ending in 0, or empty: a composition of $n - 1$. (c_{n-1} of these.)

Bijection 2, for $c_{n+1} = f_n$, i.e. $C(z) - 1 = z\, F(z)$

Use the stars-and-bars argument: a composition of $n + 1$ is a choice of which of n gaps to cut; write 1 for an uncut gap, 0 for a cut. Parts of size ≤ 2 means no two consecutive uncut gaps, i.e. no 11.

(c) and (d), for the record

(c) $K(z) = \frac{1}{3}\Big[\frac{1}{1-2z} + \frac{2}{1+z}\Big] = \frac{1-z}{(1-2z)(1+z)} = \frac{1-z}{1-z-2z^2}.$

Matrix side: $\det(I - zA) = 1 - 3z^2 - 2z^3 = (1-2z)(1+z)^2$ (eigenvalues $2, -1, -1$), and the $(1,1)$ cofactor is $1 - z^2$, so

$$\big[(I - zA)^{-1}\big]_{11} = \frac{(1-z)(1+z)}{(1-2z)(1+z)^2} = \frac{1-z}{(1-2z)(1+z)}. \qquad \checkmark$$

(d) $1/P(z) = 1 - z - z^2 - 3z^3 - 13z^4 - 71z^5 - 461z^6 - \cdots$

OEIS A003319: $1, 1, 3, 13, 71, 461, \ldots$ counts **indecomposable permutations** (no proper prefix $\sigma(1)\cdots\sigma(k)$ that is a permutation of $[k]$). Let $D(z) = z + z^2 + 3z^3 + 13z^4 + \cdots$ be their OGF. Every permutation is uniquely a sequence of indecomposable ones, so $P(z) = 1 + D + D^2 + \cdots = \frac{1}{1 - D(z)}$: an identity in $\mathbb{C}[[z]]$ between two series that converge nowhere.

Before you go

- **Reading for Monday:** Flajolet–Sedgewick §I.2, pages 24–39 (the symbolic method).
- **Problem set 3** is due tonight at 10 pm; submission details on the course page.
- Kyle is back Monday.

Today in one line

A class is a set with sizes; its OGF is the class with each object replaced by z^{size}; $\mathbb{C}[[z]]$ makes every identity between such series a counting statement, no convergence required.