

- Goals
- Orthonormality
 - Gram-Schmidt algorithm

$(V, \langle \cdot, \cdot \rangle)$ an inner product space over $F = \mathbb{R}$ or $\mathbb{C}$. $u \perp v$

Defn For $S \subseteq V$, S is orthogonal if $\langle u, v \rangle = 0 \quad \forall u \neq v \in S$;

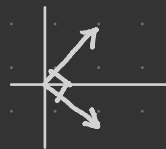
it is orthonormal when, additionally, $\langle u, u \rangle = 1 \quad \forall u \in S$.

E.g. ① $\{e_1, \dots, e_n\}$ is orthonormal in F^n $\|u\| = 1$

② $\left\{ \frac{1}{\sqrt{2}}(1, 1), \frac{1}{\sqrt{2}}(1, -1) \right\}$ is orthonormal in F^2 .

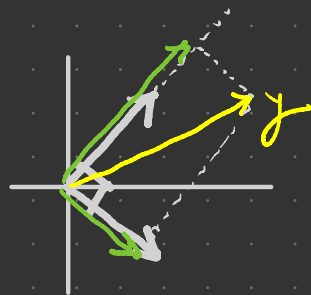
③ $\{2 \sin(2\pi x), 2 \cos(2\pi x)\}$ is orthonormal in $C_{\mathbb{R}}[0, 1]$

$$\int_0^1 (2 \sin(2\pi x))^2 dx = \int_0^1 (2 \cos(2\pi x))^2 dx = 1, \quad \int_0^1 2 \sin(2\pi x) \cdot 2 \cos(2\pi x) dx = 0$$



Prop Let $S = \{v_1, \dots, v_k\} \subseteq V$ be orthogonal. Then for $y \in \text{span } S$,

$$y = \sum_{j=1}^k \underbrace{\frac{\langle y, v_j \rangle}{\|v_j\|^2}}_{\text{proj'ns of } y \text{ onto } v_j} v_j.$$



Pf Say $y = \sum_{i=1}^k \lambda_i v_i$. Then

$$\langle y, v_j \rangle = \left\langle \sum_i \lambda_i v_i, v_j \right\rangle$$

$$= \sum_i \lambda_i \langle v_i, v_j \rangle \quad [\langle \cdot, \cdot \rangle \text{ linear in 1st var}]$$

$$= \lambda_j \langle v_j, v_j \rangle \quad [\langle v_i, v_j \rangle = 0 \text{ if } i \neq j]$$

$$= \lambda_j \|v_j\|^2.$$

Thus $\lambda_j = \frac{\langle y, v_j \rangle}{\|v_j\|^2} \Rightarrow y = \sum_j \frac{\langle y, v_j \rangle}{\|v_j\|^2} v_j$. $\square$

Cor If $S \subseteq V$ is orthonormal and $y \in \text{span } S$, then

$$y = \sum_{j=1}^k \langle y, v_j \rangle v_j \quad \square \quad (\|v_j\|^2 = 1)$$

Cor If $S \subseteq V$ is an orthogonal set of vectors, then S is linearly independent.

WTS: $\lambda_i = 0$

Pf Let $S = \{v_1, \dots, v_n\}$ and suppose $\sum \lambda_i v_i = 0$. Then

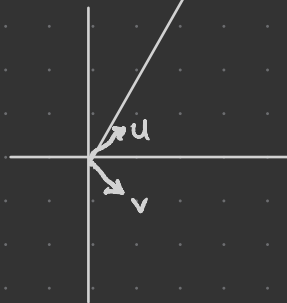
$$0 = \langle 0, v_j \rangle = \langle \sum_i \lambda_i v_i, v_j \rangle = \sum_i \lambda_i \langle v_i, v_j \rangle = \lambda_j \langle v_j, v_j \rangle$$

Since $\langle v_j, v_j \rangle \neq 0$, we have $\lambda_j = 0$. $\square$

E.g. Let $\beta = \left(\overset{u}{\frac{1}{\sqrt{2}}}(1,1), \overset{v}{\frac{1}{\sqrt{2}}}(1,-1) \right)$, an orthonormal basis for $\mathbb{R}^2$.

Let $y = (4,7)$. What are the β -coordinates of y ?

$$\begin{aligned} \text{We have } y &= \langle y, u \rangle u + \langle y, v \rangle v \\ &= (4,7) \cdot \left(\frac{1}{\sqrt{2}}(1,1) \right) u + (4,7) \cdot \left(\frac{1}{\sqrt{2}}(1,-1) \right) v \\ &= \frac{11}{\sqrt{2}} u - \frac{3}{\sqrt{2}} v. \end{aligned}$$



Goal Transform a linearly independent set into an orthonormal set with the same span.

Algorithm [Gram-Schmidt]

Input: Lin ind set $S = \{w_1, \dots, w_n\} \subseteq V$.

(1) Let $v_1 = w_1$.

(2, ..., n) For $k=2, \dots, n$ define

$$v_k = w_k - \sum_{i=1}^{k-1} \frac{\langle w_k, v_i \rangle}{\|v_i\|^2} v_i \dots$$

Think of as "straightening" w_k wrt $v_1, \dots, v_{k-1}$ by removing proj's onto them

Output: $S' = \{v_1, \dots, v_n\} \subseteq V$ orthogonal with $\text{span } S' = \text{span } S$.

Output: $S'' = \left\{ \frac{v_1}{\|v_1\|}, \dots, \frac{v_n}{\|v_n\|} \right\} \subseteq V$ orthonormal with $\text{span } S'' = \text{span } S$.

$$\{(1,2), (0,-1)\}$$

Validity Pf [Idea] Use induction to check that the "straightening" operation preserves span and induces orthogonality.
(Full details below.) $\square$

Cor Every finite dimensional inner product space has an orthonormal basis. $\square$

E.g. Take $V = \mathbb{R}[x]_{\leq 1}$ with inner product $\langle f, g \rangle = \int_0^1 f \cdot g$.

Let's apply Gram-Schmidt to $\{1, x\}$:

$$(1) \quad v_1 = 1$$

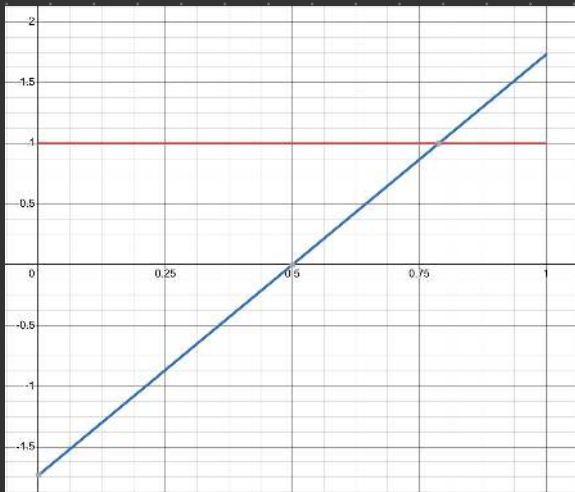
$$(2) \quad v_2 = x - \frac{\langle x, v_1 \rangle}{\|v_1\|^2} v_1 = x - \frac{\langle x, 1 \rangle}{\|1\|^2} 1$$

$$= x - \int_0^1 x \, dx \cdot 1 = x - \frac{1}{2}.$$

Normalizing: $\|v_1\| = \sqrt{\int_0^1 1^2 \, dx} = 1.$

$$\|v_2\| = \sqrt{\int_0^1 (x - 1/2)^2 \, dx} = \sqrt{1/12}$$

So $\{1, \sqrt{12}(x - 1/2)\}$ is an orthonormal basis of $\mathbb{R}[x]_{\leq 1}.$



Q How can we interpret orthonormality of these visually?

Problem Apply 6.5 to $\{(1, 2), (0, -1)\}$ in $\mathbb{R}^2.$

A

$$v_1 = (1, 2)$$

$$v_2 = (0, -1) - \frac{(0, -1) \cdot (1, 2)}{\|(1, 2)\|^2} (1, 2)$$

$$= (0, -1) - \frac{-2}{5} (1, 2)$$

$$= (0, -1) + \left(\frac{2}{5}, \frac{4}{5}\right)$$

$$= \left(\frac{2}{5}, -\frac{1}{5}\right)$$

check: $(1, 2) \cdot \left(\frac{2}{5}, -\frac{1}{5}\right) = \frac{2}{5} + 2\left(-\frac{1}{5}\right) = 0$

$$\text{So } S' = \left\{ (1, 2), \left(\frac{2}{5}, -\frac{1}{5}\right) \right\}$$

$$\left\| \left(\frac{2}{5}, -\frac{1}{5}\right) \right\| = \sqrt{\frac{5}{25}} = \frac{\sqrt{5}}{5}$$

$$S'' = \left\{ \frac{1}{\sqrt{5}} (1, 2), \frac{\sqrt{5}}{5} (2, -1) \right\}$$

Proof of validity of the algorithm. We prove this by induction on n . The case $n = 1$ is clear. Suppose the algorithm works for some $n \geq 1$. and let $S = \{w_1, \dots, w_{n+1}\}$ be a linearly independent set. By induction, running the algorithm on the first n vectors in S produces orthogonal $v_1, \dots, v_n$ with

$$\text{Span}\{v_1, \dots, v_n\} = \text{Span}\{w_1, \dots, w_n\}.$$

Running the algorithm further produces

$$v_{n+1} = w_{n+1} - \sum_{i=1}^n \frac{\langle w_{n+1}, v_i \rangle}{\|v_i\|^2} v_i.$$

It cannot be that $v_{n+1} = 0$, since otherwise, the above equation we would say

$$w_{n+1} \in \text{Span}\{v_1, \dots, v_n\} = \text{Span}\{w_1, \dots, w_n\},$$

contradicting the assumption of the linear independence of the w_i . So $v_{n+1} \neq 0$.

We now check that v_{n+1} is orthogonal to the previous v_i . For $j = 1, \dots, n$, we have

$$\begin{aligned} \langle v_{n+1}, v_j \rangle &= \left\langle w_{n+1} - \sum_{i=1}^n \frac{\langle w_{n+1}, v_i \rangle}{\|v_i\|^2} v_i, v_j \right\rangle \\ &= \langle w_{n+1}, v_j \rangle - \sum_{i=1}^n \frac{\langle w_{n+1}, v_i \rangle}{\|v_i\|^2} \langle v_i, v_j \rangle \\ &= \langle w_{n+1}, v_j \rangle - \frac{\langle w_{n+1}, v_j \rangle}{\|v_j\|^2} \langle v_j, v_j \rangle \\ &= \langle w_{n+1}, v_j \rangle - \langle w_{n+1}, v_j \rangle \\ &= 0. \end{aligned}$$

We have shown $\{v_1, \dots, v_{n+1}\}$ is an orthogonal set of vectors, and we would now like to show that its span is the span of $\{w_1, \dots, w_{n+1}\}$. First, since $\{v_1, \dots, v_{n+1}\}$ is orthogonal, it's linearly independent. Next, we have

$$\text{Span}\{v_1, \dots, v_{n+1}\} \subseteq \text{Span}\{v_1, \dots, v_n, w_{n+1}\} \subseteq \text{Span}\{w_1, \dots, w_n, w_{n+1}\}.$$

Since $\text{Span}\{v_1, \dots, v_{n+1}\}$ is an $(n+1)$ -dimensional subspace of the $(n+1)$ -dimensional space $\text{Span}\{w_1, \dots, w_n, w_{n+1}\}$, these spaces must be equal. $\square$