

- Goal
- Best fit problems
 - Least squares solutions
 - Two ways to find least squares solutions

Today $F = \mathbb{R}$, $\mathbb{R}^n$ carries dot product.

Defn For $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, a least squares solution of $Ax = b$ is a vector $\hat{x} \in \mathbb{R}^n$ such that

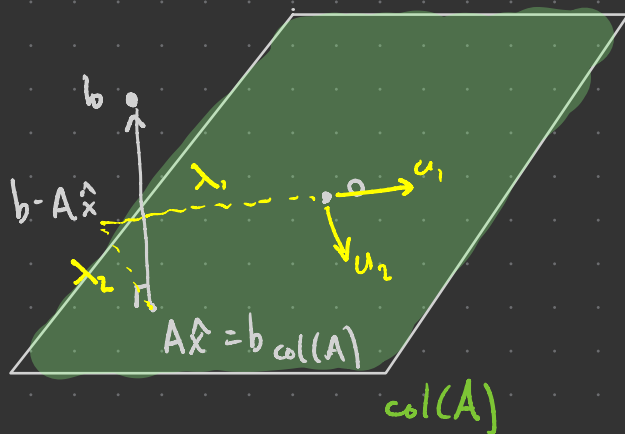
$$\|b - A\hat{x}\| \leq \|b - Ax\|$$

for all $x \in \mathbb{R}^n$.

Note $\{Ax \mid x \in \mathbb{R}^n\} = \text{col}(A)$, the column space of A .

So $A\hat{x} = b_{\text{col}(A)}$, the orthogonal proj'n of b onto $\text{col}(A)$.

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{A} & \mathbb{R}^m \\ \hat{x} & \mapsto & A\hat{x} \end{array} \quad \begin{array}{l} \text{min's dist} \\ \text{from } b \end{array}$$



$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$A\hat{x} = \lambda_1 u_1 + \lambda_2 u_2$$

The entries of $\hat{x}$ are the coordinates $\Rightarrow \text{Rep}_{\{u_1, u_2\}} \hat{x} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$ of $b_{\text{col}(A)}$ wrt columns of A when the cols are lin ind.

Thm A vector $\hat{x} \in \mathbb{R}^n$ is the least squares solution of $Ax = b$ iff it's a solution of the associated normal system $A^T A x = A^T b$.

Pf idea $\overset{\text{Norm of}}{r(x)}$ is minimized when it is orthogonal to the row space of A .

Show that $\text{row}(A)^\perp = \ker(A^T)$ so $\hat{x}$ satisfies

$$A^T r(\hat{x}) = 0 \iff A^T(b - A\hat{x}) = 0$$

$$\iff A^T A \hat{x} = A^T b.$$

Fact The normal system $A^T A x = A^T b$ has a unique sol'n iff $\ker A = 0$
iff columns of A are lin ind.

Algorithm For $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

① Compute $A^T A$, $A^T b$.

② Row reduce $[A^T A \mid A^T b]$

③ This is always consistent and any sol'n $\hat{x}$ is a least squares sol'n.

Note When $Ax = b$ has a unique least squares sol'n, it is

$$\hat{x} = (A^T A)^{-1} A^T b.$$

$$\begin{matrix} \mathbb{R}^{m \times m} & \mathbb{R}^{m \times n} \end{matrix}$$

Now suppose the columns of A are orthogonal, say

$$A = \begin{pmatrix} | & | & \dots & | \\ u_1 & u_2 & \dots & u_n \\ | & | & \dots & | \end{pmatrix}.$$

Then $\text{col}(A)$ has orthogonal basis $u_1, \dots, u_n$ and

$$b_{\text{col}(A)} = \frac{\langle b, u_1 \rangle}{\|u_1\|^2} u_1 + \dots + \frac{\langle b, u_n \rangle}{\|u_n\|^2} u_n = A \underbrace{\begin{pmatrix} \langle b, u_1 \rangle / \|u_1\|^2 \\ \vdots \\ \langle b, u_n \rangle / \|u_n\|^2 \end{pmatrix}}_{\hat{x}}$$

E.g. (best fit line) Suppose we have points

$(0, 6), (1, 0), (2, 0)$

What line best fits this data?

Eq'n of a line: $y = Mx + B$.

"Want":

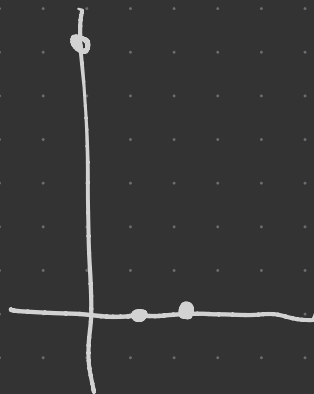
$$6 = M \cdot 0 + B$$

$$0 = M \cdot 1 + B$$

$$0 = M \cdot 2 + B$$

$$\text{Set } A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad x = \begin{pmatrix} M \\ B \end{pmatrix} \quad b = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$$

Then we "want" $Ax = b$.



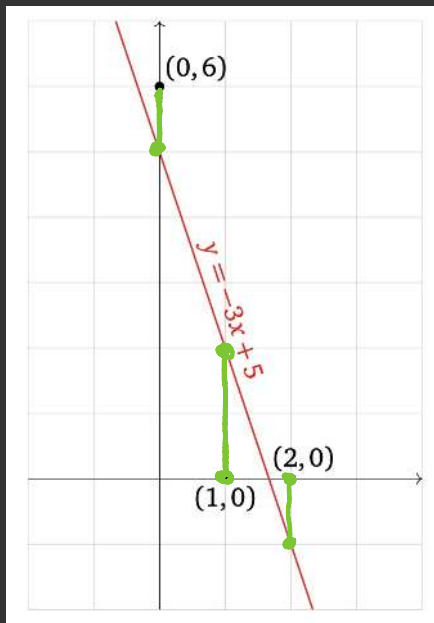
Use the algorithm:

$$A^T A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix}$$

$$A^T b = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

$$\text{and } [A^T A \mid A^T b] = \left(\begin{array}{cc|c} 5 & 3 & 0 \\ 3 & 3 & 6 \end{array} \right) \xrightarrow{G-J} \left(\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 5 \end{array} \right)$$

Thus $\hat{x} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$ is the unique least squares sol'n, and thus the best fit line is $y = -3x + 5$.



Q What has least squares minimized
in finding this best fit line?

A $A\hat{x} = \begin{pmatrix} f(0) \\ f(1) \\ f(2) \end{pmatrix}$ and $b - A\hat{x} = \begin{pmatrix} 6 - f(0) \\ 0 - f(1) \\ 0 - f(2) \end{pmatrix}$

so minimizing

$$f(t) = \hat{M}x + \hat{B}$$

$$\hat{x} = \begin{pmatrix} \hat{M} \\ \hat{B} \end{pmatrix}$$

$$\|b - A\hat{x}\| = \sqrt{(6 - f(0))^2 + (0 - f(1))^2 + (0 - f(2))^2}$$



minimizing $\|b - A\hat{x}\|^2 = \dots$