24. XI. 13

Goals
- Define inner product spaces
- Norm & length in inner product spaces

Motivation Add notions of length & angles to $\mathbb{R}$- and $\mathbb{C}$-vector spaces.

Defn Let $F = \mathbb{R}$ or $\mathbb{C}$, V an F-vs. An *inner product* on V is a function $\langle\ ,\ \rangle : V \times V \longrightarrow F$

$$(x, y) \longmapsto \langle x, y \rangle$$ ~ \langle x, y \rangle

such that

1. linear in first variable: $\langle x + \lambda y, z \rangle = \langle x, z \rangle + \lambda \langle y, z \rangle$
2. conjugate symmetric: $\overline{\langle x, y \rangle} = \langle y, x \rangle$.
3. positive definite: $\langle x, x \rangle \in \mathbb{R}_{\geq 0}$ and $\langle x, x \rangle = 0$ iff $x = 0$.

$\overline{a + bi} = a - bi$

Note

- $F = \mathbb{R}$: nondegenerate positive definite symmetric bilinear form
- $F = \mathbb{C}$: nondegenerate Hermitian form

E.g. (a) The ordinary dot product on $\mathbb{R}^n$:

$$\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle = x \cdot y = \sum_{i=1}^{n} x_i y_i$$

e.g. $\langle (1,2), (3,4) \rangle = 1 \cdot 3 + 2 \cdot 4 = 11$

$(a+bi)(a-bi) = a^2 + b^2 = |a+bi|^2$

(b) The ordinary inner product on $\mathbb{C}^n$:

$$\langle x, y \rangle = x \cdot \bar{y} = \sum_{i=1}^{n} x_i \bar{y}_i .$$

Note: $\langle x, x \rangle = \sum_{i=1}^{n} x_i \bar{x}_i = \sum_{i=1}^{n} |x_i|^2 \geq 0$

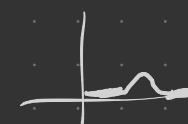
e.g. $\langle (1+i, 1-i), (1+2i, 4) \rangle = (1+i)(1-2i) + (1-i) \cdot 4$

$= 7 - 5i$.

(c) Let $V = C_{\mathbb{R}}([0,1]) = \{f : [0,1] \to \mathbb{R} \mid f \text{ cts}\}$.

Define $\langle f, g \rangle = \int_0^1 f(t)g(t)\, dt$.

For positive definiteness, note $f \neq 0 \Rightarrow f^2 \geq 0$ and $f(t)^2 > 0$ on some open interval $\Rightarrow \langle f, f \rangle = \int_0^1 f^2 > 0$.



(d) $V = \mathbb{R}^2$ and

$$\langle (x_1, x_2), (y_1, y_2) \rangle = 3x_1y_1 + 2x_1y_2 + 2x_2y_1 + 4x_2y_2 .$$

For positive definiteness, note that

$$\langle (x_1, x_2), (x_1, x_2) \rangle = 3x_1^2 + 4x_1x_2 + 4x_2^2$$

$$= 3\left((x_1 + \tfrac{2}{3}x_2)^2 + \tfrac{8}{9}x_2^2\right).$$

(e) $V = F^{m\times n}$. For $A \in V$ define the conjugate transpose of A by $A^* = \bar{A}^T$ with $A^*_{ij} = \overline{A_{ji}}$.

Then $\langle A, B\rangle := \operatorname{tr}(B^* A)$ is an inner product on V.

Check (1) $m=1$ recovers ordinary inner products on $\mathbb{R}^n$, $\mathbb{C}^n$

(2) $\langle\,,\rangle$ is positive definite.

Prop For $(V, \langle\,,\rangle)$ an inner product space over F,

① $\langle x, y+z\rangle = \langle x,y\rangle + \langle x,z\rangle$

② $\langle x, \lambda y\rangle = \bar{\lambda}\langle x,y\rangle$

$\langle\,,\rangle$ conjugate linear in 2nd variable

$\langle\,,\rangle$ is sesquilinearity

1.5

③ $\langle x, 0\rangle = \langle 0, y\rangle = 0$

④ If $\langle x,y\rangle = \langle x,z\rangle$ $\forall x \in V$, then $y = z$.

<u>Pf</u> ① $\langle x, y+z\rangle = \overline{\langle y+z, x\rangle} = \overline{\langle y, x\rangle} + \overline{\langle z, x\rangle}$

$= \langle x,y\rangle + \langle x,z\rangle$.

②, ③ : Left to you :)

④ If $\langle x,y\rangle = \langle x,z\rangle$ for all x then

$0 = \langle x,y\rangle - \langle x,z\rangle$

$= \langle x,y\rangle + \overline{(-1)}\langle x,z\rangle$

$= \langle x,y\rangle + \langle x,-z\rangle$ [by ②]

$= \langle x, y-z \rangle$ for all x. [by ①]

In particular, for $x = y-z$ we get $0 = \langle y-z, y-z \rangle$

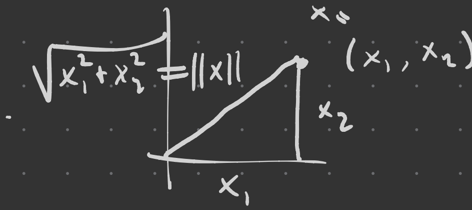
$\Rightarrow \underset{\text{[pos def]}}{y-z=0} \Rightarrow y = z$. $\square$

<u>Defn</u>. Let V be an inner product space. The <u>norm</u> or <u>length</u> of $x \in V$ is $\|x\| := \sqrt{\langle x, x \rangle}$.

- Two vectors $v, w \in V$ are <u>orthogonal</u> or <u>perpendicular</u> when $\langle v, w \rangle = 0$.
- A <u>unit vector</u> is $v \in V$ such that $\|v\| = 1 \iff \langle v, v \rangle = 1$.

E.g. (a) $V = \mathbb{R}^n$, $\langle x, y \rangle = x \cdot y$ then

$$\|x\| = \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$



(b) $V = \mathbb{C}^n$, $\langle x, y \rangle = x \cdot \bar{y}$ then

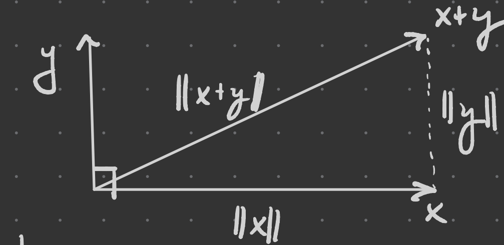
$$\|z\| = \sqrt{z \cdot \bar{z}} = \sqrt{\sum_{i=1}^{n} z_i \bar{z}_i} = \sqrt{\sum_{i=1}^{n} |z_i|^2}.$$

If we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via $x + iy \longmapsto (x, y)$ then this matches the norm on $\mathbb{R}^{2n}$.

(c) $V = C_{\mathbb{R}}[0,1]$ then $\|f\| = \sqrt{\int_0^1 f^2}$, ($L_2$-norm)

Thm [Pythagoras redux] Let $(V, \langle\,,\rangle)$ be an inner product space and let $x, y \in V$ be orthogonal. Then

$$\|x\|^2 + \|y\|^2 = \|x+y\|^2 .$$



Pf We know $\langle x, y\rangle = 0$, so $\langle y, x\rangle = \overline{\langle x, y\rangle}$
$= \bar{0} = 0$ too.

Thus $\|x+y\|^2 = \langle x+y, x+y\rangle$

$= \langle x,x\rangle + \cancelto{0}{\langle x,y\rangle} + \cancelto{0}{\langle y,x\rangle} + \langle y,y\rangle$ [sesquilinearity]

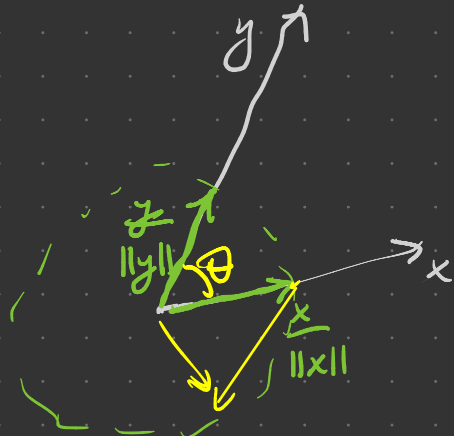
$= \|x\|^2 + \|y\|^2 .$ $\square$

Question Now know about "angle measurements of $\frac{\pi}{2}$":

$x \perp y \iff \langle x, y\rangle = 0 .$

orthogonal

How should we define $\angle(x,y)$ in general?
(arbitrary $x,y \in V$)



$$\langle x,y\rangle = \|x\|\|y\|\cos\theta$$

$$x,y\neq 0 \Rightarrow \cos\theta = \frac{\langle x,y\rangle}{\|x\|\|y\|}$$

$$\Rightarrow \theta = \arccos\left(\frac{\langle x,y\rangle}{\|x\|\|y\|}\right)$$

for $F=\mathbb{R}$

$$A^{-1}A = I_{n+1}$$

$$A^n = \begin{pmatrix} F & \cdots \\ \cdots & \cdots \end{pmatrix} \qquad A=\begin{pmatrix}0&1\\1&1\end{pmatrix}$$