

- Goals
- Powers of diagonalizable matrices
 - Graphs via matrices
 - Matrix powers and counting walks on graphs.

Suppose $A \in F^{n \times n}$ is diagonalizable. Then $A = P^{-1}DP$ for $D = \text{diag}(\lambda_1, \dots, \lambda_n)$, $P \in GL_n(F) = \{Q \in F^{n \times n} \mid Q \text{ invertible}\}$

Powers of A ?

$$\begin{aligned}
 A^2 &= (P^{-1}DP)(P^{-1}DP) \\
 &= P^{-1}D(P P^{-1})DP \\
 &= P^{-1}D^2P = P^{-1} \text{diag}(\lambda_1^2, \dots, \lambda_n^2)P
 \end{aligned}$$

In general,

$$A^k = P^{-1} D^k P$$

$$= P^{-1} \text{diag}(\lambda_1^k, \dots, \lambda_n^k) P \quad \text{by induction.}$$

Q What about non-diag'le matrices? $P^{-1} J^k P$,
J in Jordan form

Graphs via matrices

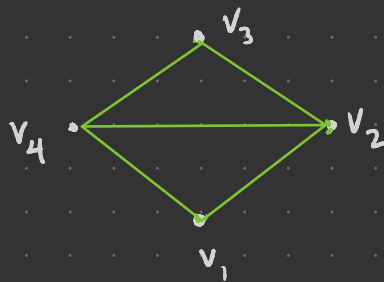
A simple graph is $G = (V, E)$, $V = \text{vertices}$

$$E \subseteq \binom{V}{2} = \{ \{v, w\} \mid v \neq w \in V \}$$

E.g. $V = \{v_1, v_2, v_3, v_4\}$

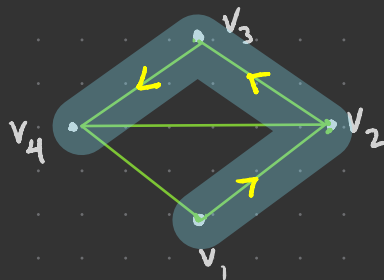
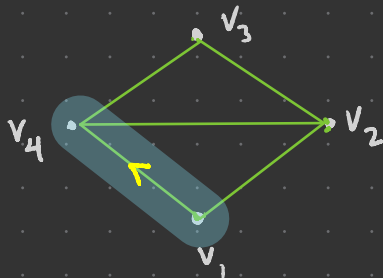
$$E = \{ \{v_1, v_2\}, \{v_1, v_4\}, \{v_2, v_4\}, \{v_2, v_3\}, \{v_3, v_4\} \}$$

Visually:



A walk (of length l) in G is a sequence of vertices $u_0 u_1 \dots u_l$ such that $\{u_i, u_{i+1}\} \in E$, $0 \leq i \leq l-1$.

E.g. In above graph, $v_1 v_4$ and $v_1 v_2 v_3 v_4$ are walks from v_1 to v_4 of lengths 1 and 3, respectively.



Defn Let $G = (V, E)$ be a graph with $V = \{v_1, \dots, v_n\}$.

The adjacency matrix of G is the $n \times n$ matrix

$A = A(G)$ with

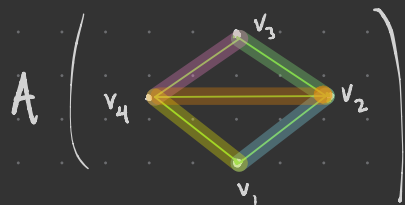
$\in \mathbb{R}^{n \times n}$

$$A_{ij} = \begin{cases} 1 \\ 0 \end{cases}$$

$\{v_i, v_j\} \in E$

$\{v_i, v_j\} \notin E$

E.g.



$$= \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix} \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Note Depends
on choice of
ordering of
 V .

Thm If $A = A(G)$, then the number of walks from v_i to v_j of length l in G is $(A^l)_{ij}$.

Pf HW! $\square$

Problem For $A = A\left(\begin{array}{c} v_3 \\ v_4 \cdot \quad \cdot v_2 \\ v_1 \end{array}\right)$ compute A^2, A^3 to

determine (a) # length 2 walks v_2 to v_3 ,

(b) # length 3 walks v_2 to v_2 .

Then find all such walks.

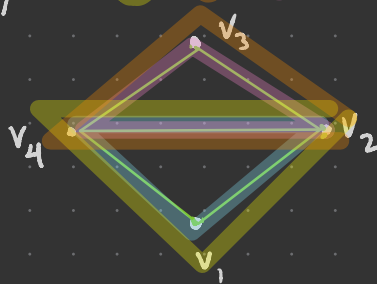
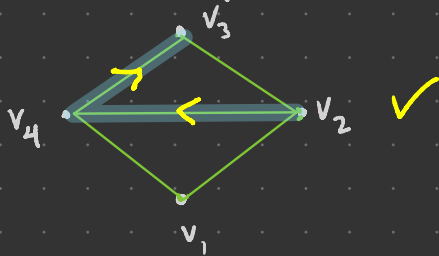
$$\begin{matrix} & v_1 & v_2 & v_3 & v_4 \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{matrix}$$

Defn A matrix A is symmetric when $A = A^T$.

Note If $A = A(G)$, then A is symmetric.

$$A^2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 5 & 2 & 5 \\ 5 & 4 & 5 & 5 \\ 2 & 5 & 2 & 5 \\ 5 & 5 & 5 & 4 \end{pmatrix}$$



Thm If A is an $n \times n$ symmetric matrix with real entries, then A is diagonalizable.

Pf This is a corollary of the "Spectral Theorem". $\square$

Upshot Adjacency matrices are diagonalizable!

Cor Given a graph G with n vertices and $0 \leq i, j \leq n$

$\exists c_1, \dots, c_n \in \mathbb{R}$ (independent of l) such that

$$\begin{array}{c} \# \text{walks } v_i \text{ to } v_j \text{ of} \\ \text{length } l \text{ in } G \end{array} \overset{\text{depend on } i, j}{=} \sum_{r=1}^n c_r \lambda_r^l$$
$$\square$$

where $\lambda_1, \dots, \lambda_n$ are eigenvalues of $A(G)$ (counted w/ multiplicity)

Idea # length l walks v_i to $v_j = (A^l)_{ij}$

$$= (P^{-1} \text{diag}(\lambda_1^l, \dots, \lambda_n^l) P)_{ij}$$

Defn A walk is closed when it starts and ends at the same vertex.

Cor The number of length l closed walks in G is $\text{tr}(A(G)^l)$. $\square$

Prop For $A \in F^{n \times n}$, $\text{tr}(A)$ = sum of eigenvalues of A counted according to algebraic multiplicity.

Note If $\chi_A(x) = c(x - \lambda_1) \cdots (x - \lambda_n)$, then $\text{tr}(A) = \sum_{i=1}^n \lambda_i$.

This sum is in F (b/c $\text{tr}(A) \in F$) even if λ_i are not!

Pf of Prop Let $\bar{F}$ = algebraic closure of F .

$\exists P \in GL_n(\bar{F})$ such that $P^{-1}AP = J$ is in Jordan form.

The diagonal of J is $\lambda_1, \dots, \lambda_n$. Now

$$\text{tr}(A) = \text{tr}(PJ P^{-1})$$

$$= \text{tr}(P P^{-1} J)$$

$$[\text{tr}(UV) = \text{tr}(VU)]$$

$$= \text{tr}(J)$$

$$= \sum_{i=1}^n \lambda_i$$

□

Cor Suppose $A(G) \in \mathbb{R}^{n \times n}$ has eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ listed with algebraic multiplicity. Then the number

of closed walks in G is $\lambda_1^l + \dots + \lambda_n^l$.

E.g. If $A = A\left(\begin{array}{c} v_3 \\ v_4 \text{ --- } v_2 \\ v_1 \end{array}\right)$, then

$$\chi_A(x) = x(x+1)(x^2 - x - 4)$$

with roots $0, -1, \frac{1+\sqrt{17}}{2}, \frac{1-\sqrt{17}}{2}$.

Thus the # closed walks in G of length l is

$$w(l) = 0^l + (-1)^l + \left(\frac{1+\sqrt{17}}{2}\right)^l + \left(\frac{1-\sqrt{17}}{2}\right)^l$$

where $0^l = \begin{cases} 1 & l=0 \\ 0 & l>0 \end{cases}$.

l	0	1	2	3	4	5	6
$w(l)$	4	0	10	12	50	100	298

spectral graph theory!